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ABSTRACT

This bulletin suggests teaching procedures for a tenth-grade geometry course. An introduction states general objectives and discusses the nature of inductive and deductive thinking. Lesson outlines follow which give a statement of objectives and a plan for development. Major lesson groupings are congruent triangles, parallel lines, quadrilaterals, coordinate geometry, circles, similarity, mean proportion, trigonometry, area, regular polygons, locus, inequalities, and solid geometry. Unit tests follow each section. (LS)

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CURRICULUM BULLETIN · 1964-65 SERIES · NUMBER 8

MATHEMATICS: TENTH YEAR

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FOREWORD

This publication is a revision of the 1959 teaching guide for 10th year mathematics. The original guide was issued to help teachers implement the mathematics 10 section of the 1957 version of the New York State Syllabus--Mathematics 10-11-12. Every teacher of 10th year mathematics should have a copy of this State Syllabus.

The aspects in which the present edition differs from the original are as follows:

1. The introduction has been rewritten.
2. The scope of content has been omitted.(For scope, see New York State Syllabus--Mathematics 10-11-12.)
3. Many alternate sequences are suggested in the text of the guide. (See also page 6 and the suggested time schedule which is there.)
4. The introductory lessons have been rewritten.
5. The initial introduction of coordinate geometry has been expanded in a separate chapter to present it as a separate postulational system.
6. The unit on circles precedes the unit on similarity.
7. A new approach to the unit on area has been developed.
8. The unit on regular polygons has been rewritten.
9. A completely revised unit on inequalities has been developed.
10. Many small errors have been corrected.
11. To increase the precision of language and thought processes, this publication refers to concepts and mathematical notation recommended by the School Mathematics Study Group (S.M.S.G.) and the University of Illinois Committee on School Mathematics (U.I.C.S.M.).
12. Provision for enrichment has been included. In particular, the last chapter suggests places and methods for integrating plane and solid geometry.

This publication is reissued to meet the continuing needs of the high schools. A program of revision is currently being carried on by the State and the City which will eventually result in new syllabus material.

JOSEPH O. LORETAN
Deputy Superintendent of Schools

July, 1964

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This publication is the result of the cooperative efforts of the Division of High Schools, Maurice D. Hopkins, Acting Associate Superintendent, and the Division of Curriculum Development, Jacob H. Shack, Acting Associate Superintendent. Production was carried out under the supervision of Seelig Lester, Assistant Superintendent in charge of mathematics, High School Division, and William H. Bristow, Assistant Superintendent, Bureau of Curriculum Research.

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This bulletin is a revision of the teaching guide for 10th year mathematics published in 1959. The members of the committee which prepared the original guide were:

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INTRODUCTION

The guide offers approaches which will clarify the tenth year part of the State Syllabus, Mathematics 10-11-12, and suggests teaching procedures for achieving the objectives of that syllabus. By the end of this year of mathematics, the student should have a clear understanding of the major methods of reasoning developed by mankind - 1) induction 2) deduction. The student should know that both the inductive and deductive methods are essential to the progress of civilization. They complement one another in helping man improve his understanding, not only of his physical world, but also of his philosophical world. The lessons in this guide remind the teacher to make the distinctions between the two types of thinking frequently and to test that understanding. The study of smaller sequences as stressed in the State Syllabus is an important step toward the better understanding of the deductive process and postulational systems.

In our K-12 mathematics courses, we are concerned with the use of deductive proof in arithmetic and algebra, as well as in geometry. The feeling for mathematical structure gained through a postulational approach to arithmetic and algebra can make them much more meaningful. However, it is still true that the study of geometry in the Tenth Year can clarify the essence of the two methods of reasoning, as they complement one another, better than any other single year of mathematics, and, if well taught, can give the student a lasting concept long after he forgets the details of various theorems.

Inductive versus Deductive Thinking

1. The Source of Propositions

In our K-9 classes, students will have reached through measurement or inspiration, many conclusions about the properties of geometric figures. These conclusions, and others arrived at quickly at the beginning of each new lesson, can now be labeled as propositions - that is, proposals for investigation. The degree of validity of these proposals depends, when only inductive thinking is being done, on the randomness of the sampling and the number of cases tried. There will be many opportunities in Tenth Year Mathematics to reinforce the understanding of the dangers of drawing conclusions from a poorly chosen sample - the common error in reasoning known as 'hasty generalization'. For example, in the initial motivating exercise of a lesson on altitudes, a student dropped a perpendicular to the base of an isosceles triangle and to the base of an equilateral triangle. His 'discovery' was, "It seems as if the altitude of any triangle bisects the opposite side."

It is suggested that, whatever the intuitive method (measuring, counting, guessing) by which a proposition is obtained by the class, the conclusion be written: "It seems as if ..."

2. Deduction

On the contrary, deduction is a process in which the validity of the conclusions does not, in any sense, depend on the number of cases investigated. The conclusions are as valid as the assumptions (postulates) from which they were deduced. The process of deduction, in which each statement is valid because the previous statement is valid, and so on back to the unproved propositions (postulates) is one which leads to confidence in the validity of one's conclusions to the degree that one has confidence in the validity of one's postulates. Students should realize, by the end of the year, that different sets of postulates lead to different sets of theorems. Each of these structures is as valid in the situation to which it is to be applied as are the postulates in that situation. Mathematics does not deal with truth in the usual sense of the word.

A distinction should be made between postulational geometry and physical geometry. It is important that students studying Mathematics 10 realize the difference between the concept of a "point" or a "line" or a "triangle" and the dot and ruler drawn "line" or triangle the student sees on a blackboard, and a surveyor uses in designing buildings or highways.

When we begin with a set of undefined terms and a set of postulates, we are considering "geometry" from a postulational point of view, and are developing an abstract mathematical system.

When we use a blackboard or notebook to make drawings of points, lines, triangles or other geometric figures, we are considering a "model" of an abstract mathematical system.

Just as the number "two" is an abstract idea, and the numeral "2" is a symbol for this idea, so a "line" is an abstract idea and a "_____" is a symbol for this abstract idea. The student should realize that the "_____" is not a "line" any more than "2" is the number "two".

In the tenth year this idea should be presented informally only, and occasionally referred to, as for example in the following situations:

> How do you know that the bisector of an angle of a triangle intersects the opposite side at a point between the other two vertices? See S.M.S.G., Geometry with Coordinates, part I, pg. 91.
>
> How do you know that the diagonals of a parallelogram intersect, and if they do intersect, how do we know they intersect inside the parallelogram?

With respect to "betweenness" the student should be told that Euclid did not consider this concept, and that this was an error of omission. The usual "proof" that all triangles are isosceles can be shown after the pupil has developed sufficient background to point up the need for consideration of order of points on a line. The consequences of disregarding the axioms on betweenness is illustrated by such famous paradoxes as the one above, and by the proof that an acute angle is equal to a right angle. And, in a bright class, it can be proved that the diagonals of a parallelogram do intersect inside the parallelogram using the parallel postulate, and postulates on betweenness.

For most classes, at least the fact that these are assumptions that we make should be emphasized. "If the diagonals --- intersect, then they bisect each other" could be a wording used once or twice to provide this emphasis. Critical thinking is an important objective of Mathematics 10. While we do not want to get involved in too much rigor, an appreciation of the deficiencies in Euclid's development of geometry is desirable.

The set of postulates used must be consistent (conclusions deduced from the postulates should not be contradictory), and should ideally be independent (no one postulate can be derived from the others). These requirements will gradually be appreciated by the students, even though in this course with these young people, it is not desirable to limit the set of postulates too rigorously. Teachers will judge the extent to which they wish to be rigorous in their effort to minimize the number of propositions assumed by their classes by the quality of the response. Any decision to 'postulate' a proposition, all year, should be made by the class keeping in mind the meaning of and reasons for using deductive thinking.

3. Why deduction?

In deductive thinking each proposition is presented for:

a) rejection, b) postulation (assume it) or c) deduction (prove it).

Why does mankind want to rearrange propositions in a deductive sequence? Students at first are just as convinced of the validity of the discovery that the sum of the angles of any triangle seems to be a straight angle, which they reached by induction, as they are after the sequence starting with parallel lines makes them able to deduce the same property. It takes patient, careful planning to reinforce the sophisticated reasons why the many ideas of man demand a rearrangement into postulational sequence. The common statements: "How do you know it is valid for the thousandth triangle?" or "You remember in Geometry we always prove everything." are not very convincing! The real idea to be put across is that deduction enables us to do two desirable things:

a) In the set of discoveries of the scientist from various types of experimental thinking, certain ideas often turn out to contradict others. These contradictions are not noticed unless the propositions are rearranged so that a few (as few as possible) are accepted as valid and all the others follow from those few - all others MUST be valid IF those few are. This idea is strengthened as we continue the course until students are anxious to deduce each new discovery from previously proven theorems instead of 'postulating' it. For enrichment, with better classes, these ideas can carry the student into stimulating research, but even with normal classes, the teacher should continually call attention to what the class is doing with various propositions. Otherwise the students lose sight of the aim of the course in the multifarious theorems developed during the year. The small sequences, stressed in the State Syllabus make this task more feasible, but the teacher will find that a question such as, "What was the postulate of our Area Sequence?" will produce an intelligent response only if the course is so taught.

b) The other reason for rearranging ideas in a postulational sequence is that many unexpected propositions are deduced from previously proven theorems and thus become theorems which would probably not have been discovered inductively. Teachers should have a wealth of examples of this fact available, such as the deduction of the existence of the planet Neptune. In plane geometry, it is unlikely that students would discover the constant product as a chord rotates about a fixed point in a circle, yet the proposition is an immediate deduction from the similarity sequence. The fact that the sum of the squares of the legs is equal to the square of the hypotenuse in any right triangle, would not be a likely discovery by students without deduction.

GENERAL TEACHING SUGGESTIONS

1. What is an "Informal" Proof?

Frequent references will be made to this kind of proof. It is a deductive proof and carries all the authority of a "formal" deductive proof. The latter is characterized by the two-column arrangement in which each statement is supported by a reason. The informal proof has an essay form in which statements are not always supported by a stated reason, particularly when that reason should be obvious. Only the crucial steps need reason support. What constitutes a crucial step will depend on the background of students' accomplishments. This will have to be a matter for the teacher's judgment.

Informal proofs are used to save the time usually devoted to either reciting or recording a proof formally. They are used only for such propositions as are not classified as "required for Regents Examination Purposes." The informal proof is particularly useful when several propositions are to be taught in one lesson, such as a set of corollaries for a theorem.

2. Time Allotments for Lessons

It will be noted that the committee suggested how the syllabus items may be organized into lessons. We caution the teacher not to be too rigid in adhering to our lesson divisions. These are only suggested and should, therefore, serve only as a guide for the teacher. Naturally, the teacher's organization of lessons will depend upon the interest and the ability of the students.

3. Student-Teacher Interchanges

In order to clearly convey some teaching suggestions, there are a number of accounts of teacher questions and student answers. These student-teacher interchanges are not intended to be a prediction of the answers which students will give. However, the importance of the interchange is in the questions asked by the teacher and should be read with this in mind.

Throughout the year the teacher should:

set up a classroom situation in which questions are encouraged
give students freedom and time to think
make clear through his questions the place of that which is being considered in relation to previous and future learnings.

4. Notebook Lists

As the content of Tenth Year Mathematics unfolds, the student will appreciate the fact that he is dealing with three types of statements - definitions, postulates, and theorems. He will learn to distinguish among the statements and, furthermore, find it necessary from time to time to refresh his memory concerning the status of a statement. For these reasons, it is advisable that the student keep a list of statements which he can consult.

A section of the student's notebook may be devoted exclusively to each of the three categories. Each statement, as it is accepted, will thus be recorded in the appropriate section. Each list may be divided into topical sections in accordance with those topics (congruence, similarity...) discussed in this publication.

An alternate method would be to record the statements under topics and identify each statement as a definition, postulate, or theorem.

As students learn that theorems furnish short cuts in deductive proofs, they will be interested in recording them as methods of deduction. When the teacher wishes to emphasize this use of a theorem, it may be done by asking students to record the theorems that could be used in deducing relations such as the following:

Methods of deducing line segments are equal
Methods of deducing angles are equal
Methods of deducing angles are supplementary
Methods of deducing lines are parallel
Methods of deducing triangles are congruent
Methods of deducing triangles are similar
Methods of deducing triangles are equal
Methods of deducing ratios are equal
Methods of deducing products are equal

It is advisable to use the following form in listing these methods:

To prove two angles equal, show that

the angles are opposite two equal sides of a triangle
the angles are inscribed in the same arc of a circle, etc.

Whenever a new method is derived, it should be added to the list.

SUGGESTED TIME SCHEDULE AND ONE OF THE POSSIBLE SEQUENCES

Topic	Approximate Number of Lessons
Introductory Lessons	8
Congruent Triangles and Constructions	17
Parallel Lines	7
Congruent Triangles Continued and Two Locus Theorems	6
Quadrilaterals	9
Introduction to Coordinate Geometry-Another Postulational System	3
Circles	16
Similarity	12
Mean Proportional and Pythagorean Theorem	11
Trigonometry and Slope	7
Area	8
Regular Polygons and Circles	8
Locus (Continued)	3
Inequalities	4
	119 *

*The 119 lessons do not include lessons for tests and review.
Note: Alternate sequences are suggested throughout. For example, area before similarity, similarity before circles, inequalities after quadrilaterals, indirect reasoning as early as feasible, coordinate geometry after Pythagorean Theorem, are indicated as possibilities, each with its own advantages.

I.

THE INTRODUCTORY LESSONS

Lesson 1

Aim: To introduce students to the purpose of teaching geometry in the tenth year. To develop an appreciation of the need for careful definitions in deductive reasoning and to show the need for undefined terms.

Development:

It is assumed that tenth year students already have some knowledge of the ordinary daily uses of geometry learned in their 7th, 8th, and 9th year mathematics classes. A brief discussion may be presented of the various needs for geometry (geo refers to earth, metry refers to measure) by surveyors, navigators, carpenters, plumbers, machinists, architects, builders, engineers, physicists and others. In addition, there are many phenomena in nature that require a knowledge of geometry in order to understand them. These include eclipses of the sun and moon, and structures of crystals, molecules, and atoms.

The teacher should inform pupils that in addition to acquiring knowledge of a body of "facts," the principal aim of the course is to teach students methods of reasoning. Geometry is the simplest and most convenient vehicle for illustrating methods of reasoning. Geometry uses a method of reasoning that is a model for philosophers, scientists, editorial writers, statesmen and other serious-minded people. This method (deductive) is used by them as a means of convincing others to share their opinions. It is the way in which good lawyers argue their cases, or the good legislators argue for their proposals. It is the method that Sherlock Holmes used to reach certain conclusions from apparently trivial facts that he observed. As an illustration of deduction, say to the class (of at least 32 pupils), "I am sure that in this class there are at least two of you whose birthday is on the same day of the month - 10th, 11th, etc." This will create an enthusiasm for deduction in the class. The teacher should reinforce this interest by mentioning some less elementary deductions, like the determinations of the weight of the earth and the discovery by deduction of the existence of the planet, Neptune.

Note that the method is important as a means of presenting proof, as well as a method for discovering it. This aspect of the deductive method should be made clear to the student throughout the year. It is not enough to find a proof. It is also important to learn how to present it.

Now have the class consider some situation such as the following: Two students get into an argument over the statement, "All fish breathe under water," one arguing that it is true and the other citing the fact that whales must surface to breathe as evidence that it is false.

Some pupils will object that whales are mammals, not fish. Elicit from the class the statement that the word 'fish' must be carefully defined before we can consider the merits of the argument. Generalize this idea to develop the notion that logical reasoning requires very careful and precise definitions of any words which are used in the reasoning process if one is to prevent misunderstanding and also errors in reasoning.

Elicit that words are defined in terms of other words. What must be true of these other words? Consider a dictionary which defines marsh as swamp, swamp as bog, and bog as marsh. Would a person not knowing the meaning of any of these three words be helped by this dictionary? Is there any way out of this dilemma? Have the class realize that although the dictionary defines a large number of words, this number is *finite,* and therefore the situation described is unavoidable unless we wish to start with some words which we assume everybody knows the meaning of. In a logical system, we *must* begin with certain undefined terms which serve as the basis for defining other terms.

What are some of the undefined terms used in geometry? Present point, line and plane to the class as basic undefined terms. Discuss the "properties" of these elements and also tell pupils how their *representations* are named *with* *letters* to distinguish one from another in a logical argument involving more than one. At this point it is essential to emphasize the difference between the abstract points and lines to be used in logical reasoning and the physical strokes and dots on papers and blackboard which are used to represent them (see the discussion of abstract geometry vs. a physical model of it which is given in the introduction to this guide). Show the class how the undefined terms can be used in defining other terms. Define line segment, ray, and polygon. Have the pupils start a page in their notebooks headed "Definitions" and record these.

The S.M.S.G. and U.I.C.S.M. courses make a distinction between a line, a line segment, and a ray (with distinct notation for each). All are defined as sets of points. They also distinguish the measure of the length of a segment, which is a positive real number from the segment itself. It is suggested that each teacher of Mathematics 10 have available to him a set of the S.M.S.G. and U.I.C.S.M. texts. These may be used to provide enrichment for incorporation into the New York State Mathematics 10 Course of Study.

> In the S.M.S.G. GW (geometry with coordinates) text, Part I, Chapter 2, pgs. 33-48 introduces the concept of sets applied to geometric figures and the incidence properties of points, lines and planes.
>
> On page 38, the notation for line AB as $\overline{AB}$ is introduced.
>
> In Chapter 3, the distinctions between line AB, ray AB and segment AB are developed - the definition of a ray appears on page 84, of a line segment on page 86 and page 88 (top) contains a summary of these.

It is not suggested that the teacher necessarily use various notations for line, line segment, ray, measure of the length of a line segment unless he wishes to, but that the class be continually made aware of the distinctions and know which they mean either by notation or by context.

The lesson may be applied by using other exercises involving the need to define words carefully. Ask the class to decide whether "Abraham Lincoln was an educated man." It will soon be seen by the difference of opinions that the term "educated" has different meanings to different members of the class. One meaning refers to formal or school training, the other refers to worldly experiences. Clearly, the need for good definitions is a necessity if we are all to hold the same meanings for words in a discussion.

Lesson 2

Aim: To illustrate the difference between induction and deduction.

Development:

Have pupils draw triangles. Obtain the midpoints of two sides by measurement, and connect them with a line segment. Measure the line segment thus formed and compare its length to the length of the third side of the triangle. Have 5 or 6 pupils do this at the board (or on acetate for overhead projector) with varying sizes and shapes of triangles. Elicit the generalization that "It seems as if the line segment joining the midpoints of the two sides of a triangle is equal to half of the third side." This is a proposition, i.e., a proposal for investigation. Is it true? Is it possible that after we have tried this experiment on a thousand triangles in which the generalization is true, we may then come across one in which it is false? Is it likely that we would find such a case? The generalization above was arrived at by a process of induction. An induction assigns a property discovered to be true in a certain subset (the triangles we tried) to the whole set (all triangles).

Now tell the class to suppose that by logical reasoning mathematicians had actually proved that it is always true that the line joining the midpoints of two sides of a triangle is equal to half the third side. Knowing this, suppose we now have a specific triangle ABC with midpoints D and E of sides AB and BC respectively and AC is known to be 10" long, how long is DE? Point out that the length of DE was arrived at by a deduction - we did not measure it! A deduction may be considered as the assignment of a property of a set to a subset. The statement above was known to be true of all triangles, and it was therefore asserted to be true of a subset consisting of the specific triangle ABC.

Note: In this experiment, we have used the concept of line segments being "equal". What does AB = CD mean? This statement means that the measure of the length of the line segments is the same number. Length of a line segment is defined in some modern texts by assigning a coordinate system to the line of which the segment is a part. Symbolism like m $(\overline{AB})$ or $|AB|$ may be used for the measure of the length of line segment AB, if the teacher desires.

Suppose John, Tony, and Carl are three boys in the class, who therefore form a subset of the class. If we note that John, Tony, and Carl are all wearing red ties and make the generalization that "All the boys in this class are wearing red ties," we are using induction - assigning a property of a subset to the whole set. On the other hand, if we happened to know that all the boys were required to wear red ties, we could state that "John, Tony and Carl were wearing red ties," without looking at them. We will then be using deduction, assigning a property of the whole set to a subset.

Students should appreciate the differences in the validity of conclusions arrived at by generalization (induction) and by deduction. Because three boys are seen wearing red ties, are we certain that all the boys are wearing red ties? Because we know that all the boys are wearing red ties, are we certain that three specific boys among them do? Discuss the problem of improving the validity of the method of generalization (induction) by

a. increasing the number of specific cases before generalizing
b. selecting a good sampling of the generalized set

Start a list of "Common Errors in Reasoning" in notebooks. The error developed here is "Improper generalization from too few cases or a poor sampling." Be sure the class sees that simply taking many cases does not insure a good sampling. For example, one might assume that water contracts when the temperature drops by taking measurements of its density at many different temperatures. However, between 4° and 0° water expands as the temperature drops.

Lesson 3

Aim: To examine the nature and requirements of a good definition. The teacher may use the definitions of terms learned in the 7th, 8th, and 9th years as the substance out of which the criteria of good definitions may be developed.

1. Definitions and Sets

Definitions may be clarified in terms of sets. The object defined must be identified as a member of a set and then must be distinguished from other members of the set. These two parts of a definition are called classifying and distinguishing.

For example: A pencil is a writing implement that has lead. Thus, a pencil is a member of the set of writing implements which contains pens, chalk, crayons, brushes, and so on. The pencil is different from these in that it alone has lead. On the other hand, a writing implement is a hand implement used for writing. Thus, the set of writing implements is itself a member of a larger set of hand implements containing hammers, crowbars, brooms, and so on. It differs from these in that it alone is used to write with.

It is now evident that definitions may form a sequence. In the above illustration, the sequence was (starting with the smallest set) pencil, writing implement, hand implements. We might extend the sequence to "implements," to "manufactured articles," to "inanimate articles," and so on.

2. Defining Geometric Terms

Have the students now undertake the formal definitions of geometric terms. Start with angle, kinds of angles, including acute, obtuse, straight, and right angles. Teachers will have various possible choices for these definitions - an angle as a rotation, an angle as the union of two rays from the same origin, etc. depending on the text used. Be sure to stress the sequence in the definitions, whichever you use.

> In Chapter 4 of the S.M.S.G. GW text, the definition of "angle" is given on page 145, and the measure of an angle developed on pages 149-154. Notice that the GW course considers as angles only those with measures between 0 and 180, excluding the "straight angle" as an angle. One of the reasons for this involves the property that the open segment joining any two points, one on each side of the angle, lies in the interior of the angle. Note too that when we talk of an angle of 30°, we mean the measure of the angle is 30 with "degrees" being assumed as the unit. On page 165, "betweenness" for rays is defined, and on pages 184-185, congruent angles are defined as angles with equal measures.

Continue the discussion of definitions by developing definitions for polygon, triangle, and kinds of triangles classified by sides and angles, using the definitions to illustrate the properties of a good definition.

It is desirable to build definitions by imbedding the term being defined in the next larger set. However, it is not incorrect to define a term by placing it within a set that is not the next immediately larger one, provided the properties necessary to distinguish it from the other members of the set are included in the definition. Thus, later in the year a rhombus can be defined as a parallelogram with two adjacent sides equal, or as a quadrilateral which is equilateral. In discussing the definitions of angles and triangles, the pupils should also be shown the accepted notation for naming them, including the use of one or three letters in naming angles and the purpose of each.

The building of the definitions of the various kinds of triangles affords an opportunity to have pupils note the nesting of sets, one within the other. This may seem to students to be an endless provess, and this can be used to reinforce the previously developed need for undefined terms.

Suggested exercises:

1. After defining "line segment," define

   median
   altitude
   angle bisector

2. Arrange the following terms to form a correct sequence for definitions starting with the largest set:

   a. equilateral triangle, polygon, triangle, isosceles triangle
   b. square, rectangle, polygon, quadrilateral
   c. hunting dogs, canines, dogs, pointers
   d. Ford, car, vehicles, Thunderbird

Have the class note that a definition is simply a statement of the equivalence between a lengthy description of a term and a simple expression for the term. 'Isosceles triangle' and 'triangle with two equal sides' mean the same thing. Ask the class what they can conclude if you "give" them a triangle ABC with AB=BC. Then ask them what they can conclude if you "give them an isosceles triangle ABC with AC as its base. Elicit that definitions are reversible.

Teach the use of the symbol ∴.

Examples:

a) 1. $\triangle$ ABC is isosceles with base AC	1. Given
2. ∴ AB=BC	2. By definition, an isosceles $\triangle$ is a $\triangle$ with 2 equal sides.

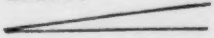
b) 1. AB=BC	1. Given
2. ∴ $\triangle$ ABC is isosceles	2. By definition, a $\triangle$ with 2 = sides is an isosceles $\triangle$.

Later in the year, since a definition is an equivalence, the teacher could permit the two steps above to be omitted from a proof and the inference to be made above the proof in the hypothesis.

Repeat the above type of exercise with other definitions, asking for a reason to support the conclusion each time. This will provide drill in the definitions and also build the groundwork for the concept of a deductive proof consisting of statements each supported by reasons.

Lesson 4

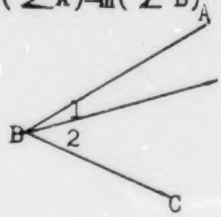
Aim: To develop the meaning of postulate and theorem and the distinction between them.

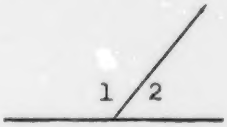
Have the class consider a diagram showing a pair of vertical angles and let them suggest a definition for the term "vertical angles" and add it to the list of definitions. What seems to be true of the relative size of the two vertical angles in a pair? Before answering this question, what do we mean by the size of an angle? Again, the different definitions of angle will require different definitions of size. We will use the symbol $\angle A = 30^\circ$ to mean the measure of the size of angle A is 30, in degree units. To clarify this, be sure students see that ∟ is a larger angle than 

What do we mean by equal angles? $\angle A = \angle B$ means that $m(\angle A) = m(\angle B)$

Discuss the meaning of $\angle 1 + \angle 2$

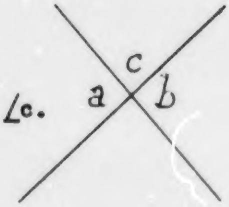
$\angle ABC - \angle 1$



Apply to 

$\angle 1 = 180^\circ - \angle 2$

How shall we investigate the proposition that, "It seems as if vertical angles are equal"? Pupils will suggest measurement with a protractor. Elicit that the generalization made as a result of such measurements is an induction and review what its limitations are. Is it possible to deduce the proposition that "If two angles are vertical angles, they are equal" for all possible pairs of vertical angles, from some simpler proposition that we are all willing to accept? How can such a deduction be made? Get the class to see that to prove propositions by a logical argument requires reasons to support the statements (or claims) made in the argument. What can these reasons be? Desirably, the reasons should be propositions which have already been proved. Those propositions were also proved from previous propositions. But how can the first of these propositions be proved? Relate this situation to that developed in connection with definitions and establish the need for propositions which we accept without proof. We call such assumed propositions postulates or axioms.



Consider the vertical angles a and b.

Now have the class note that we can express $\angle a$ as $180^\circ - \angle c$.

We can also express $\angle b$ as $180^\circ - \angle c$.

Angle a and angle b can't both equal $180^\circ - \angle c$ unless they are equal, but we need a reason to support the statement that $\angle a = \angle b$.

Get the class to see that we really assumed that "Quantities equal to the same quantity are equal to each other" and that this is our first postulate. Have pupils begin their notebook list of postulates with this entry using a heading, "Postulates we have assumed." Recall that such postulates were also used in algebra although they may not have been called by this name (perhaps pupils knew them as "principles" they did not prove).

For example, remind the students of the commutative postulate of arithmetic, which states that $ab=ba$ for any natural numbers a and b.

In algebra, they probably did some deductions also, such as the proof that $a \cdot 0 = 0$ using the distributive postulate and the definition of "0", namely, for every number b, $b + 0 = b$.

Have the class now see that acceptance of our new postulate makes it possible to prove our proposition, "If two angles are vertical, then they are equal". A proved proposition is called a theorem. Have the class list this first theorem in their notebooks, labeled "Theorems we have proved." Since this theorem is a proved proposition, it can be used in future as a reason to support other deductions, so we now have one postulate and one theorem with which to make further deductions. The theorem and the postulate should now be applied in very simple deductions which they make possible. Supply the class with a fact, and challenge them to "deduce" another fact from it. Then challenge them to give a reason to support each fact or statement they deduce. (This should be done orally at this stage, not written out formally.)

What are acceptable as possible reasons in this game of deduction? Certainly, postulates (which have been accepted without proof) and theorems (which have been proved) may fill this role. Show the class that some facts must also be supplied at the beginning of each logical argument and if these are cited in a proof, the reason supporting them is stated as "given." Next, ask the class to tell you what they concluded about $\triangle$ ABC if it was given that it is isosceles. Discussion should reveal that definitions may also serve as reasons for supporting statements made in a deductive proof, giving us four possible types of reasons.

Note to Teacher:

The teacher may now introduce the concept of a postulational system as the organization of undefined terms, definitions, postulates and theorems. It may be pointed out that the year's project in Tenth Year Mathematics is the construction of such a postulational system concerning the materials of geometry. However, one may also build a postulational system with the materials of algebra or any of the branches of mathematics. Moreover, scientists, philosophers, and others who want to organize their ideas systematically strive to use a postulational system as their model.

It is interesting to note that Einstein's Relativity Theory was built partly on a denial of Newton's postulate that space and time were absolute (independent of any observer), while Einstein did postulate that the speed of light was absolute. The time for discussing non-Euclidean Geometry is better postponed to the lesson when the parallel postulate is discussed.

Other examples may be presented which bear close resemblance to a postulational system. The Declaration of Independence may be such a one. It postulates early in the document that "We hold these truths to be self-evident: That all men are created equal: that they are endowed by their Creator with certain inalienable rights;..." After presenting a mass of evidence it concludes (theorems) with, "We, therefore...declare, that the united colonies are, and of right out to be free and independent states..." Concepts, such as "free," "equal" are left undefined, whereas "inalienable rights" is defined in terms of "life, liberty, and the pursuit of happiness."

Lessons 5 and 6

Aim: To develop more definitions and to use them in building the students' concept of simple deductions.

Now that we have defined one type of angle pair (vertical angles), let us turn to definitions of other common angle pairs. Adjacent, supplementary, and complementary angles should have definitions developed and recorded. An appreciation of the need for precise definition should be developed through such questions as, "Are three 30° angles complementary?" Diagrams of overlapping angles with a common side and a common vertex can similarly be used to emphasize the need for such a detail as the requirement that the common side be between them in the definition of adjacent angles.

Now have the class consider two equal angles, each of which has a supplement. What can we conclude about the supplements? They will use the postulate of the previous lesson to prove that "If two equal angles each have a supplement, then the supplements are equal." Is this a postulate or a theorem?

Similarly, the theorem, "If two equal angles each have a complement, then the complements are equal" should be developed and listed in the notebooks. Note the advisability, wherever possible, of writing propositions in the "if-then" form. Apply these theorems in simple deductions, asking the class for oral conclusions (statements) each time and a reason to support each statement.

Consider a situation involving more than one right angle or more than one straight angle. Get the class to suggest and prove the theorems, "If two angles are right angles, then they are equal" and "If two angles are straight angles, then they are equal." Use these in making further deductions in situations involving right angles or straight angles. It may also be desirable to introduce and use another postulate, "A quantity may be substituted for an equal quantity in an equation" at this point. The teacher may point out that this postulate was used throughout ninth year mathematics whenever a variable was replaced by an equal expression.

Lessons 7 and 8

Aim: To provide further experience with deductive proofs including some involving more than one step, some requiring identification of what is given and what is to be proved, and some requiring diagrams to be made by the student.

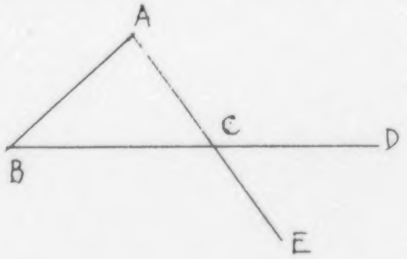
Ask the class to draw two lines which are perpendicular to each other. They will usually draw one vertical and one horizontal. Challenge them to decide whether two lines not so oriented can be perpendicular. What is the essential feature that determines whether lines are perpendicular? Evolve the definition that "Perpendicular lines are lines that meet at right angles." Apply this definition to deductions concerning the angles formed when two lines are given to be perpendicular, and to deductions concerning the lines when the angles at which they meet are right angles, using also the theorem that "If two angles are right angles, then they are equal."

Multi-step deductions: Have the class note that the definition of perpendicular lines permit us to deduce that we have right angles. What can we then deduce concerning these angles? Note that often a deduction leads us to a fact which in turn permits a second deduction and so on. Thus, a whole chain of deductions may result from one set of given facts.

Note: The approach of this guide makes these early deductions easier because the students have only a limited number of postulates to work with.

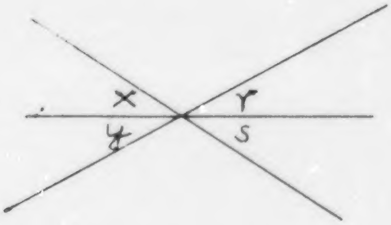
a. Given: $\angle ABC = \angle ACB$
all lines are straight.

Deduce: $\angle ABC = \angle DCE$



b. Given: $\angle x = \angle y$
all lines are straight

Deduce: $\angle r = \angle s$



The teacher should encourage the following form of argument.

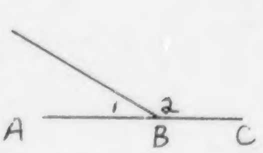
Steps: $x = y$

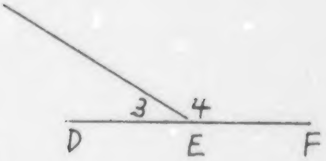
But $x = s$
$\therefore$ $y = s$

But $y = r$
$\therefore$ $s = r$

c. Given: $\angle 1 = \angle 3$
ABC and DEF are straight lines

Deduce: $\angle 2 = \angle 4$



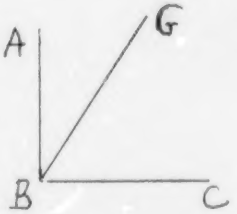


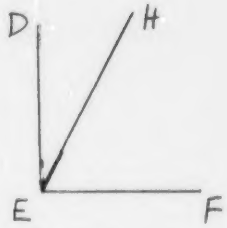
Note: This uses the theorem on supplements which was deduced from our single postulate. This is a good time to introduce the meaning of sequence in a postulational system.

d. Given: $AB \perp BC$, $DE \perp EF$

$\angle ABG = \angle DEH$

Deduce: $\angle GBC = \angle HEF$





More exercises involving "two-deduction" proofs may be found in any textbook.

The "if-then" statement: Have students consider the theorem, "If two angles are right angles, then they are equal." What information does the statement assert is given concerning these angles? What information concerning the angles does this statement enable us to prove? At this point students may be presented with the concept of statements of implication which contain a hypothesis and a conclusion. The "if-then" form of these statements is useful in identifying the hypothesis as the "if" clause and the conclusion as the "then" clause. It should be made clear that there is no judgment concerning the truth of the hypothesis or the conclusion. The implication asserts that if or whenever the hypothesis is known to be true, then the conclusion will also be true.

Suggested exercises:

Have the students write in the "if-then" form and state the hypothesis (given) and the conclusion (to prove):

Iron expands when it is heated.

All straight angles are equal.

Nylon sweaters are warm.

Supplements of equal angles are equal.

Complements of equal angles are equal.

The medians to the legs of an isosceles triangle are equal.

When two angles are equal, their supplements are equal.

Two angles are equal when their complements are equal.

When it rains, the sidewalks are wet.

Diagrams: Consider the proposition , "The bisector of the vertex angle of an isosceles triangle bisects the base." Students find two difficulties here: 1) They cannot tell the hypothesis from the conclusion in a simple sentence, 2) They cannot visualize the situation when a diagram is not given. They should suggest rewording the proposition in the "if-then" form and defining such terms as bisector, vertex angle, and base of an isosceles triangle. Bring out the desirability of having a diagram to picture the proposition. Have the class draw a diagram to illustrate it and letter the diagram. The need for a large clear diagram and uniform lettering to facilitate class discussion should be brought out. The teacher should circulate about the room to inspect students' notebooks. State the hypothesis (given) in terms of the lettered diagram and also the conclusion (to prove) in terms of the lettered diagram.

Suggested exercises:

Make a lettered diagram, state the hypothesis (given) and the conclusion (to prove) in terms of the diagram for each of the geometric statements in the earlier exercises in this lesson.

UNIT TEST

THE INTRODUCTORY LESSONS

1. a. Arrange these terms so that each is a subset of the term which follows:

   polygon, geometric figure, isosceles triangle, triangle, equilateral triangle

   b. State the superset that is used in the definition of each of the following:

   1) median of a triangle
   2) vertical angles
   3) acute angle
   4) scalene triangle
   5) supplementary angles

   c. Complete the following definitions:

   1) Adjacent angles are a pair of angles which...
   2) A right triangle is a triangle which...
   3) Perpendicular lines are two lines which...

2. Prove or disprove by an experiment, using two samples:

   The bisector of the obtuse angle of an obtuse triangle bisects the opposite side.

3. In each of the following pairs of statements, state which is the generalization of the other:

   a. 1) The acute angles of an isosceles right triangle are equal.
      2) The base angles of an isosceles triangle are equal.

   b. 1) If the sum of two equal angles is 90°, they are complementary.
      2) If the sum of two angles is 90°, they are complementary.

   c. 1) The sides of a triangle are segments of straight lines.
      2) The sides of a polygon are segments of straight lines.

4. Write out a chain of deductions (with reasons for each step) to prove that $\angle ABC = \angle CBD$, given that $CB \perp AD$.

5. State the hypothesis and conclusion in each of the following:

   a. If two lines intersect, each pair of vertical angles are equal.
   b. If equals are doubled, the results are equal.
   c. The corresponding parts of congruent triangles are equal.

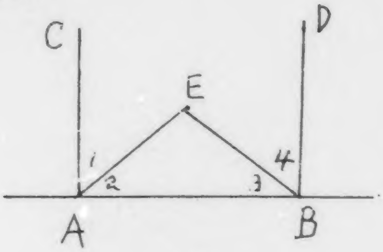
6. State a reason which justifies each conclusion:

   a. Given: $CA \perp AB$

      $DB \perp AB$

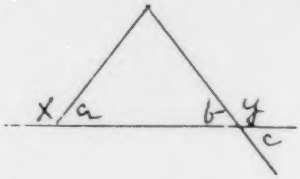
      $\angle 2 = \angle 3$

      Conclusion: $\angle 1 = \angle 4$



   b. Given: $\angle x = \angle y$

      Conclusion: (1) $\angle a = \angle b$
      (2) $\angle b = \angle c$
      (3) $\angle a = \angle c$



7. State whether each of the following is a definition, postulate or theorem:

   a. An isosceles triangle is a triangle which has two equal sides.
   b. Every angle has a bisector.
   c. Right angles are equal.
   d. Complements of equal angles are equal.
   e. If equals are subtracted from equals the remainders are equal.

II.

CONGRUENT TRIANGLES

Lesson 1

Aim: To introduce the meaning of congruent figures and congruent triangles, to present the first congruence postulate, and to apply this congruence postulate in simple proofs.

Development:

The teacher should inform the class that congruence is a basic concept used in industry. Assembly line methods of production depend upon congruent parts. Congruence is also an important logical tool, for if we know, for example, that two triangles are congruent, we can use this knowledge to deduce that any of the corresponding parts are equal.

The teacher should develop the definition of congruent figures as "Figures that agree in all of their corresponding parts." Exhibit to the class pairs of congruent polygons and pairs of congruent triangles cut out of cardboard to emphasize the concept. The advantage of the "correspondence concept" of congruence over the definition of congruence which speaks of "figures which can be made to coincide" is that the first concept avoids the need to move figures "without changing size or shape" to make them coincide. Later on, teachers may want to have pupils move figures rigidly to bring out relations not easily seen otherwise. Many of the modern courses in geometry recommend emphasizing the correspondence idea in congruence by using a method of naming congruent triangles which takes account of this; in this method of naming triangles, if it is stated that triangle ABC is congruent to triangle DEF, it is implied that A corresponds to D, B to E, and C to F and that all the corresponding parts are equal. The teacher may wish to avoid presenting a conclusion using this method of naming triangles in order to avoid giving too great a hint, but students should be encouraged to name the triangles in that order as soon as they know which parts are equal.

The teacher should have the class tell how many pairs of equal parts are necessary, according to the definition, to establish that two triangles are congruent. Have pupils list by name the eight pairs of equals for two congruent quadrilaterals and do the same for the six pairs of equals in two congruent triangles.

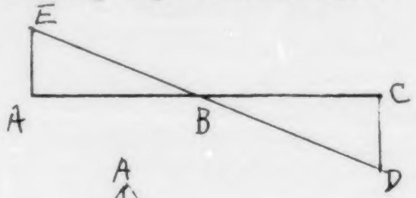
Now draw a triangle and challenge the class to construct a congruent triangle, using ruler and protractor. It will become apparent that when only some of the pairs of corresponding parts have been constructed to be equal, the remaining pairs are forced to be equal. Use the blackboard or overhead projector to have some pupils draw one triangle which is oriented differently (say upside down) from the other to disabuse pupils of some common misconceptions concerning correspondence. After reproducing the triangle by copying only some of the parts, have students verify the congruence by measuring the other corresponding parts.

In the process above, the students may have discovered any or all of the three congruence principles. The teacher will probably want to have the class concentrate for the rest of the lesson on one of the ways (say S.A.S. = S.A.S.) to force triangles to be congruent by using fewer pairs of equal parts than required by the definition of congruence. Through questioning, the teacher should elicit from the students that a principle has been discovered which will make it easier to tell whether two triangles are congruent, that the advantage of having this easier way would make them want to incorporate this principle somehow into the body of knowledge being developed, that the principle was established only by induction (some measurements), and that therefore it must be listed as a postulate. List the statement as "Two triangles are congruent if they agree in two sides and an included angle" (note that this is the correct way to have students read orally the written abbreviation S.A.S. = S.A.S.). We may now use S.A.S. = S.A.S. to deduce that triangles are congruent.

The rest of this lesson may be devoted almost entirely to writing deduction proofs in two-column form. The problems should be very simple and so stated that the "Given," "To deduce," and the diagram are given to the student. Such problems as the following might be suitable:

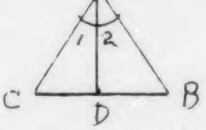
1. Given: EB = BD
   AB = BC

   To deduce: $\triangle ABE = \triangle CBD$



2. Given: $\angle 1 = \angle 2$
   AC = AB

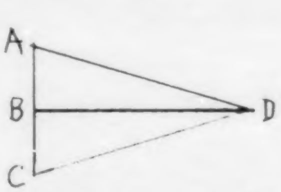
   To deduce: $\triangle ADC \cong \triangle ADB$



In attacking problem 2, students will note that they need AD = AD to establish a pair of equal sides. Use this need to introduce the postulate, "A quantity equals itself", "identity" for short, which should be added to the students' notebook list.

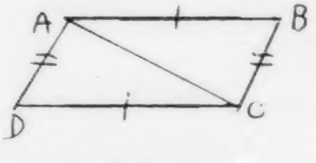
3. Given: $\angle ABD$ and $\angle CBD$ are rt. $\angle$s
   AB = BC

   To deduce: $\triangle ABD \cong \triangle CBD$



4. Given: $\angle 1 = \angle 2$
   AD = BC

   To deduce: $\triangle ABC \cong \triangle ADC$



Other equally simple problems may be found in textbooks.

It is advisable that three practices be started in this lesson which will be valuable to the student:

to mark the diagram so that he can easily see the hypothesis

to identify the statement in the first column that has established one of the desired conditions for congruence by the symbol (A) or (S); thus in problem 2, the student writes the statements:

1. AC = AB (S) (preferably in this order so that SAS will stand out)
2. $\angle 1 = \angle 2$ (A)
3. AD = AD (S)

Lesson 2

Aim: To discover and postulate the A.S.A. = A.S.A. and S.S.S. = S.S.S. congruence principles, and to apply them in simple proofs.

Procedure:

Ask the class what other methods than S.A.S. they have used to construct copies of triangles, that is, triangles congruent to given triangles. It is not necessary to repeat the experiments unless the class fails to recall the use of A.S.A. and S.S.S.

Postulate these new congruence principles and apply them in exercises similar to those found in Lesson 1. Conditions in these problems should be limited to those involving "identity", pairs of right angles, and pairs of vertical angles.

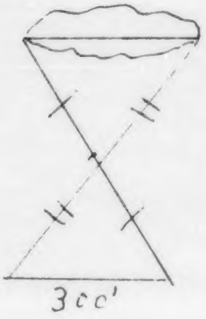
In addition to the two practices recommended in Lesson 1, have the students decide which congruence principle to use before writing each proof.

Lesson 3

Aim: To teach that corresponding parts of congruent triangles are equal as a consequence of the definition of congruence.

Development:

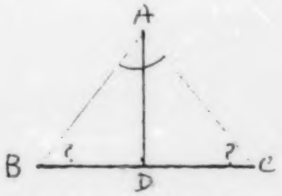
Challenge the class to find the distance across the lake from the measurements shown on the diagram. How do we know that the distance across the lake is equal to the corresponding side in the other triangle? Have students recall that the definition of congruent triangles implies that all pairs of corresponding parts are equal. Therefore, if we can first deduce that the triangles are congruent, we will then be able to deduce the equality of the desired pairs of corresponding parts.



It may be advisable to use the term "bisector of an angle" in the hypothesis of some exercises. The following exercise also helps the student to maintain the practices given in the previous lessons.

Given: AD bisects $\angle BAC$
$AB = AC$

To deduce: $\angle B = \angle C$



Notice the use of the question marks to identify the parts to be proved equal.

A question-answer interchange concerning the exercise might be as follows:

Teacher: What method can we use to prove that $\angle B = \angle C$?
Student: The method of corresponding parts of congruent triangles.
Teacher: What triangles can we use?
Student: $\triangle ABD$ and $\triangle CAD$
Teacher: What method can we use to prove that $\triangle ABD \cong \triangle CAD$?
Student: SAS
Teacher: Name a pair of sides we can prove equal.
Student: BA and AC
Teacher: Name a pair of angles we can prove equal.
Student: $\angle BAD$ and $\angle CAD$.
Teacher: Name a second pair of sides we can prove equal.
Student: AD and AD

There should be several students participating in the above interchange. At this point, one student should organize the proof while others write it in notebooks.

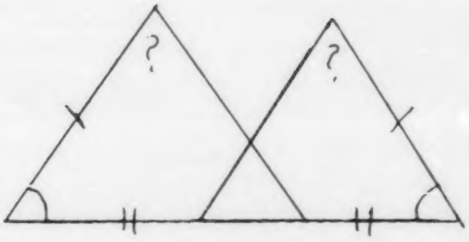
Have class do a number of similar exercises involving corresponding parts of congruent triangles.

Lessons 4 and 5:

Aim: To develop the addition and subtraction postulates and to relate them to congruence.

Development:

Challenge the students to prove this exercise:



Students will discover that two of the equal line segments are not long enough to be sides of triangles. Raise the question of what must be done to each of these line segments to get the required sides of the triangles. Students should realize that we need the postulate, "If equals are added to equals, the sums are equal."

A similar development may be used for "If equals are subtracted from equals, the remainders are equal."

The addition and subtraction axioms should now be applied apart from triangles. They may be presented

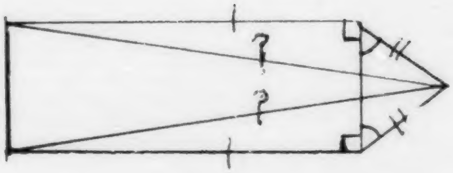
as applied to instances involving ages and money
as applied to numerical values
as applied to line segments and angles not in congruent triangles

When the postulates are understood, they should be applied to angles and sides of triangles. When incorporating these postulates within a complete proof, the teacher should make clear that three statements are needed to reach the conclusion:

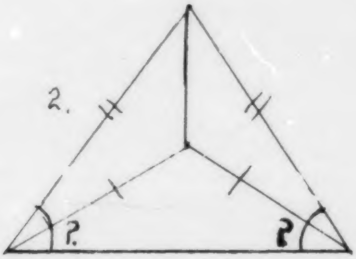
a statement of one set of equals
a statement of a second set of equals
a statement that one sum (or difference) equals another

Suggested exercises:

1.



2.



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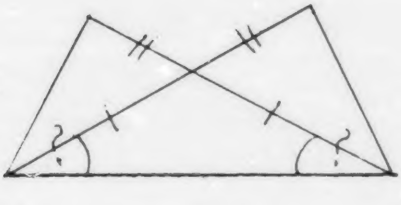

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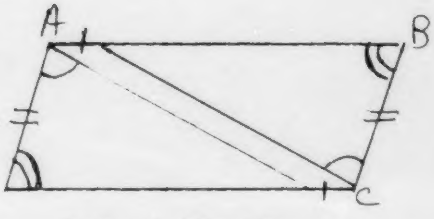


Deduce: AB = CD

Lesson 6

Aim: To relate congruence to pairs of supplementary and complementary angles

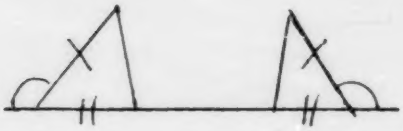
Development:

Supplementary and complementary angles have been studied and need only to be reviewed.

The three practices of earlier lessons of marking the diagram, deciding upon the congruence principle to be used, and identifying the statements which establish the conditions for congruence are to be maintained.

Suggested exercises:

1.



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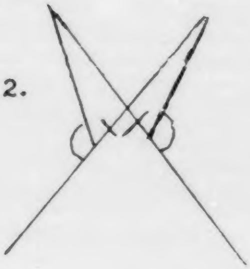

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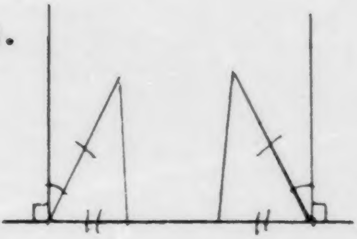

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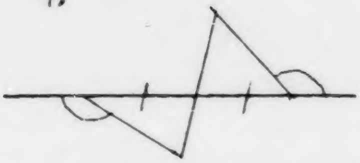

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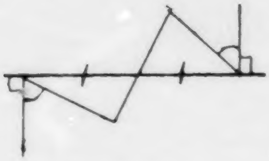

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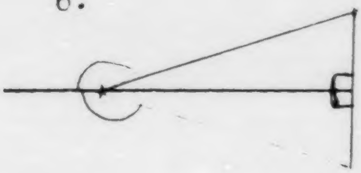


Lesson 7

Aim: To define the nature of constructions
To give the topic its setting in the history of mathematics
To discuss some of the postulates related to constructions
To teach the first construction - the bisection of an angle

Development:

As in any other new topic, students should appreciate the need for a definition. In this topic it will be necessary to define a construction as a drawing (the superset) which is limited to the use of two instruments, the compasses and a straight edge or unmarked ruler. Other drawings may be done with such instruments as the T-square, a marked ruler, parallel rules, and pantograph. It is natural to ask why this limitation has been imposed by mathematicians. The answer is to be found in the fact that assuming certain postulates of Euclidean geometry is equivalent to assuming that we can make the particular constructions which are possible with straight edge and compasses.

The postulates relating to constructions should be presented in terms of the instrument that accomplishes it. The following relate to the use of a straight edge:

> One, and only one, straight line may be drawn between two given points.
> Through a given point any number of straight lines may be drawn.
> A straight line may be extended indefinitely.

The compasses have a dual purpose. They are used to draw an arc or to lay off given lengths on lines.

The following postulates relate to the use of the compasses in constructions:

> A given line segment may be laid off any number of times on a given line.
> A circle or arc may be drawn with any given point as center and any given line segment as radius.

When presenting the bisection of a given angle, it should be listed as a construction theorem to be used to effect other constructions in the same sense as a theorem is used as the basis for subsequent deductions. It should be listed as follows:

> Construction theorem #1: To bisect a given angle.

Students should see that the construction actually involves two congruent triangles (based on the SSS principle) in which the desired halves are corresponding angles of congruent triangles. This insight is the new aspect of this construction since the construction routine has been taught in previous grades.

In teaching each new construction theorem this need to see the deduction basis for the construction should be the major emphasis in teaching and learning.

An exercise on bisecting a straight angle leads to the erection of a perpendicular to a given line at a point on the line. This latter may be elevated to the status of a construction theorem.

Other exercises may consist of constructions of angles having 45°, $22\frac{1}{2}^\circ$, 135°, and $67\frac{1}{2}^\circ$. Complements or supplements of given angles should also be constructed.

Lesson 8

Aim: To teach the duplication of given angles.
To apply this to constructing multiples, sums, and differences of angles.

Development:

Construction theorem #2, duplication of a given angle, is based upon the SSS principle as was the first construction theorem. As the steps in the construction are taught, the teacher should point out the equality of parts resulting in SSS. If this is done carefully, the student will be able to supply the deduction that proves the equality of angles.

Exercises may now be undertaken in which twice an angle, three times an angle, and so on, may be constructed when the angle is given. Exercises involving the sum or difference of angles may be tried.

Problems such as constructing $\frac{1}{2}\angle x + \angle y$ or $\frac{1}{2}(\angle x + \angle y)$ should be discussed by students in terms of the differences in order of operations.

Lessons 9 and 10

Aim: To prove the theorem - if two sides of a triangle are equal, the angles opposite these sides are equal.
To initiate a "How to deduce..." list in the student's notebook.
To give practice in the use of the theorem and congruent triangles.

Development:

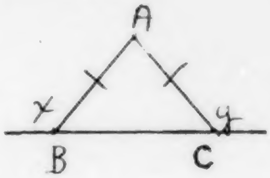
This is the first lesson in which students are asked to do a formal deduction when given only a verbal statement. It is advisable to review the concepts of hypothesis and conclusion and their identification in a sentence having the "if-then" form. When this is done and the "Given," "To deduce" and the diagram have been written, the students should have an opportunity to plan the deduction in the same manner as indicated in lesson 3. The formal deduction should now be written.

The theorem should be included in the list of theorems in the student's notebook. The student should review the significance of a theorem in a postulational system.

After the theorem is proved, accepted and recorded, students may show a tendency to repeat its proof when confronted with an exercise. They are not always aware that a theorem can act as a short cut to a conclusion when its hypothesis is known. For example,

Given: AB = AC

To deduce: $\angle x = \angle y$



Some students will draw the bisector of angle BAC and reproduce the statements and reasons of the isosceles triangle theorem in order to arrive at the conclusion that $\angle ABC = \angle ACB$. A special point must be made of the function of an established theorem as the short cut which enables the student to say that $\angle ABC = \angle ACB$ immediately after the statement that AB = AC. Thus, this theorem acts as a method of deducing that two angles of a triangle are equal when the opposite sides are equal.

The above approach should be the starting point for examining other methods of proving desired conclusions. Such an examination should lead to the establishment of a new set of notes to be called, "How to deduce..." lists. The "How to deduce that angles are equal" list will now include the following:

They are identical angles.
They are vertical angles.
They are right angles.
They are complements of equal angles.
They are supplements of equal angles.
They are the sums of two pairs of equal angles.
They are the differences of two pairs of equal angles.
They are corresponding angles of congruent triangles.
They are opposite equal sides of a triangle.

The next list, "How to deduce that line segments are equal" will contain:

They are identical line segments.
They are sums of two pairs of equal line segments.
They are differences of two pairs of equal line segments.
They are corresponding parts of congruent triangles.

The students should now do a number of deduction exercises using the isosceles triangle theorem and congruent triangles. Encourage these practices:

Mark the diagram to show the hypothesis.
Use question marks to indicate the parts to be proved equal.
Plan the proof before writing using the "How to..." lists.
Use the symbols (A) and (S) as explained in lesson 2, preferably in the order ASA or SAS.

Note: The theorem - if two sides of a triangle are equal, the angles opposite these sides are equal - may be called for on the Regents examination. It is the first of 15 such theorems. Nevertheless, it would be a mistake to inform students of this fact at this time. This knowledge may give rise to attitudes which would do harm to a student's proper understanding of the subject.

Have students realize that the "How to deduce..." list will continue to grow throughout the term. For this reason, additional space should be provided in the student's notebook.

Lesson 11

Aim: To introduce and apply the "halves of equals" and "doubles of equals" postulates.

Development:

Many students do not have a precise understanding of "halves." They equate "half" to "part." The difference between these concepts should be demonstrated with parts of angles and parts of line segments. It is sometimes effective to first compare the two halves of a single angle or of a single line segment with each other. Then compare the halves of unequal angles or of unequal line segments with each other. Only then should the class consider the halves of equals. This consideration should apply to a pair of equals as they exist independently and then as they are found as parts of the sides or angles of triangles.

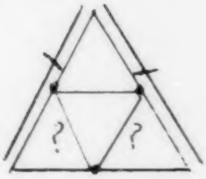
When incorporating this postulate within a complete congruence proof, the teacher should make clear that two preliminary statements are needed before the desired conclusion can be made.

A statement of equals.
A statement that each has halves.
The statement that a half of one of the equals is equal to one of the halves of the other.

Suggested exercises:

• indicates a midpoint and a broken line indicates an angle bisector.

1.



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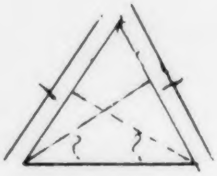

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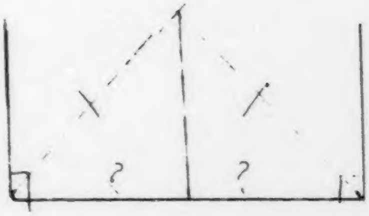

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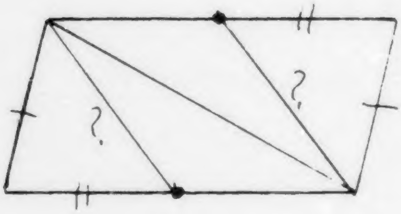


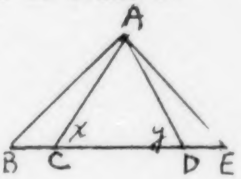
5. The trisectors of the vertex angle of an isosceles triangle are equal.

Lesson 12

Aim: To teach the student how to check a deduction proof.

Development:

Review the meaning of generalization.
Have the student do a deduction proof as follows:



Given: AB = AE
BC = DE

To deduce: $\angle x = \angle y$

Statements	Reasons
1. AB = AE (S)	1. Given
2. $\angle B = \angle E$ (A)	2. If two sides of a triangle are equal, then the angles opposite these sides are equal.
3. BC = DE (S)	3. Given
4. $\triangle ABC \cong \triangle ADE$	4. If SAS = SAS, then the triangles are congruent.
5. AC = AD	5. If triangles are congruent, then the corresponding parts are equal.
6. $\angle x = \angle y$	6. If two sides of a triangle are equal, then the angles opposite these sides are equal.

Have the student check the proof. The check consists of two parts:

1. Each statement is supported by a reason which is a generalization of the statement or "given." The reasons are definitions, postulates, theorems, or given.

2. The conditions in the hypothesis of each reason in the if-then form must have occurred as the conclusion in a previous reason, or "given." For example, the "if" clause of reason 6 has "two sides of a triangle are equal." This occurs as the conclusion of reason 5 in which "corresponding sides are equal." (The corresponding sides refer to the two sides of a triangle.)

   The check would continue with an examination of the "if" clause of reason 5 which has "two triangles are congruent." Note that this is the conclusion of reason 4.

   The "if" clause of reason 4 has three conditions S = S, A = A, and S = S. These are found to be conclusions of previous reason or referred to as "given."

One of the major values of the check is that it furnishes the teacher with a means of explaining a number of errors commonly made by students. Some of these errors are:

> The reason given does not fit the statement. This failure to fit is made clear to the student by explaining that the reason is not a generalization or that the statement is not a specific case of the reason.

Statements are omitted. The student is led to recognize this when he fails to find a hypothesis condition in a reason as the conclusion of a previous reason.

Statements are irrelevant. The student is led to see this irrelevancy when he discovers that the conclusion of a reason is not found to be an hypothesis of a subsequent reason.

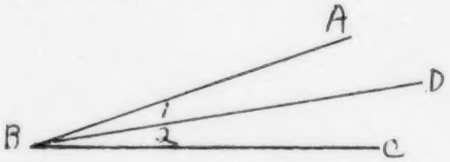
Lesson 13

Aim: To introduce the concept of "definitional equivalence". To use this concept in simplifying proofs.

Development:

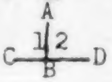
Have the student consider the following two statements:

BD bisects $\angle ABC$
$\angle 1 = \angle 2$



Since "bisect" is defined as the dividing into two equal parts, the two statements are equivalent. The two statements refer to the same information or data. Therefore, when the hypothesis of a deduction exercise contains the information "bisector of an angle," we may express this information as "$\angle 1 = \angle 2$" in the hypothesis (if $\angle 1$ and $\angle 2$ refer to the halves).

Other equivalent statements are:

(M is the midpoint of AB) $\longleftrightarrow$ (AM = MB)
($AB \perp BC$) $\longleftrightarrow$ ($\angle ABC$ is a right angle)
or $AB \perp CD$  $\longleftrightarrow$ ($\angle 1$ and $\angle 2$ are right angles)

(BD is a median of triangle ABC) $\longleftrightarrow$ (AD = DC)
(ABC is an isosceles triangle) $\longleftrightarrow$ (AB = AC)
(AD is an altitude of triangle ABC) $\longleftrightarrow$ ($\angle 1$ and $\angle 2$ are right angles)

Students should be cautioned that this equivalence is not to be confused with deductions. For example, if triangle ABC is given to be an isosceles triangle, the statement $\angle B = \angle C$ is a deduction and needs a reason to support it. However, AB = AC is definitionally equivalent and can be written in the hypothesis as such instead of "△ ABC is isosceles."

Suggested exercises: State the hypothesis and conclusion for each of the following by using the principle of "definitional equivalence."

The median to the base of an isosceles triangle bisects the vertex angle.
The median to the legs of an isosceles triangle are equal.
The angle bisectors of the base angles of an isosceles triangle are equal.
The lines joining the midpoints of the sides of an isosceles triangle form another isosceles triangle.

Lesson 14

Aim: To consider some exercises related to the isosceles triangle in order to provide more drill in proving overlapping triangles congruent, to provide more experience with medians, angle bisectors and altitudes (needed in the subsequent lessons).

Development:

Have the class prove that the medians to the legs of an isosceles triangle are equal. Get pupils to see that they may use either the lower overlapping triangles involving the base as an identity, or the upper overlapping triangles involving the vertex angle as an identity (pupils are often surprised to discover an angle used in an identity). Note the value of using colored chalk to outline the overlapping triangles, and to make the identity stand out.

Next, have the class prove that the angle bisectors of the base angles of an isosceles triangle are equal. The suggested approach and comments on the example on medians are apropos here.

Ask the class to prove that the altitudes to the legs of an isosceles triangle are equal. Why can't this be done (at this time)?

Lesson 15

Aim: To show the need for "auxiliary triangles" in some congruence proofs.

Development:

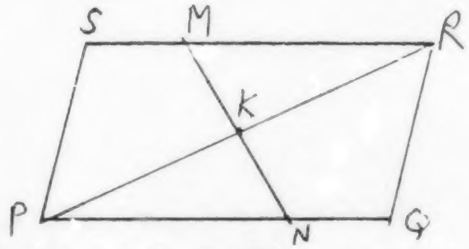
Present an exercise such as

Given: SP = QR

SR = PQ

PK = KR

To deduce: MK = KN



Students will sense the need to prove triangle MKR congruent to triangle NKP. Discussion will reveal that triangle MKR cannot be shown congruent to triangle NKP unless we can first get another pair of equal parts. Such equal parts are obtainable only by first proving triangle PSR congruent to triangle RQP.

This is a good lesson in which to stress the need for a plan before any steps are written in any proof. Teachers should also be sure to discourage listing a given fact before it is needed in a proof.

The use of colored chalk to outline pairs of congruent triangles is highly desirable in cases where various pairs of triangles under consideration overlap, as they do in this example.

Lesson 16

Aim: To develop and list ways to prove lines perpendicular.
To develop and prove the construction of the perpendicular bisector of a given line segment.

Development:

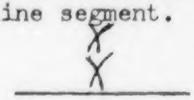
If the diagram is given, students will suggest the problem, 'Is the bisector of the vertex angle of an isosceles triangle perpendicular to the base?' Create the need for proving lines perpendicular, and start the list of, "Methods of deducing that lines are perpendicular."

The first method will arise fairly quickly - show two adjacent angles are equal and supplementary. (Prove informally and list as a theorem 'If two adjacent angles are both equal and supplementary, then the angles are right angles.')

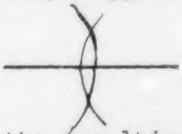
Show that there are an infinite number of perpendiculars to a given line segment at various points on it, and review the method of erecting such a perpendicular at a given point on the line as a special case of bisecting an angle.

Now how could we get the perpendicular to go through the midpoint if the midpoint were not given? Note that there are also an infinite number of lines which bisect a given line segment. The one we want is called the perpendicular bisector.

Challenge the class to discover inductively, in a few minutes, the compass and straight edge construction of the perpendicular bisector of a line segment. Students will readily suggest such methods as:



or the usual



Whichever they like best, prove that the resulting line is indeed a perpendicular bisector. The proof will involve a review of the first method of getting lines perpendicular and also the need for auxiliary triangles in congruent proofs.

Rephrase as "Two points, each equidistant from the ends of a line segment determine the perpendicular bisector of that line segment." Add to methods of deducing that lines are perpendicular: Show that you have 2 points each equidistant from the ends of a line segment.

In exercises, be sure students realize the value of this method as a great shortcut, and that they know which is the line segment and which the line determined by the 2 points.

Lesson 17

Aim: To construct angle bisectors, perpendicular bisectors of sides, medians, and altitudes in triangles; to develop the construction of dropping a perpendicular from a given point to a given line.

Development:

Have the class construct the three angle bisectors in a triangle. Also have them construct the three perpendicular bisectors of the sides of a triangle.

Next, ask the class to construct the three medians. They will see the need for the perpendicular bisectors to locate the midpoint of each side.

Finally, challenge the class to construct the three altitudes of an acute triangle. It will be realized that we have constructed a perpendicular to a line only in the case in which we could start from a point on the line. Now we are faced with the problem of constructing a perpendicular from a point outside the line. Recall the theorem that "The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to the base." Show how this theorem can be used as the deductive basis for the construction of "dropping a perpendicular."

Altitude of a triangle should be carefully defined and the cases of the altitude falling inside, outside or on the sides of the triangle discussed informally. Students should be required to construct altitudes for both acute and obtuse triangles, either in class or for homework.

Students will enjoy discovering the concurrency of the sets of three angle bisectors, three medians, three perpendicular bisectors of sides, and three altitudes. It will also be interesting for them to discuss under what special circumstances one or more of the various special lines will be identical with other special lines.

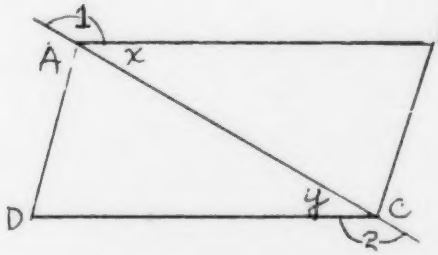
UNIT TEST

CONGRUENT TRIANGLES

1. Given: AB = DC

   $\angle 1 = \angle 2$

   Deduce: $\angle B = \angle D$



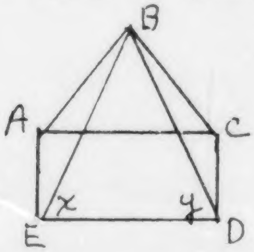
2. Read this proof carefully and be prepared to answer the questions which follow:

   Given: AB = BC

   $\angle$CAE and $\angle$ACD are rt. $\angle$s

   AE = CD

   To deduce: $\angle x = \angle y$



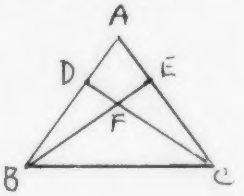
Statements	Reasons
1. AB = BC (S)	1. Given
2. $\angle CAE$ and $\angle ACD$ are rt. $\angle$s	2. Given
3. $\angle CAE = \angle ACD$	3.
4. $\angle BAE = \angle BCD$ (A)	4. If equals are added to equals, the sums are equal.
5. AE = CD (S)	5. Given
6. $\triangle ABE \cong \triangle CBD$	6. If ASA = ASA, the triangles are congruent.
7. BE = BD	7.
8. $\angle x = \angle y$	8. If triangles are congruent, corresponding parts are equal.

Questions:

1. Supply the missing reason for 3.
2. Supply the missing reason for 7.
3. A statement was omitted before statement 4. Supply it.
4. Criticize reason 6 and give the correct reason. Reason 8.
5. May statement 5 be written before statement 1? Explain.
6. May statement 3 be written after statement 4? Explain.

3. Given: AB = AC
   BF = FC

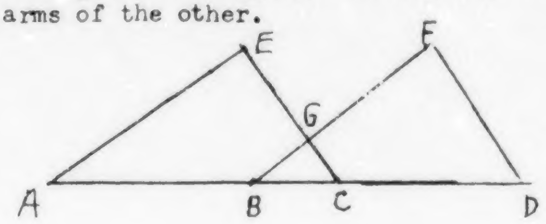
   Deduce: BD = EC



4. Prove by deduction: Two right triangles are congruent if the arms of one are equal respectively to the arms of the other.

5. Given: AB = CD
   BG = CG
   $\angle A = \angle D$

   Deduce: AE = FD



UNIT TEST

CONSTRUCTIONS

1. What instruments are permitted in geometric constructions?

2. Write one construction postulate related to the use of a straight edge.

3. Write one construction postulate related to the use of the compasses.

4. Name the principles used to deduce the correctness of the

bisection of an angle
bisection of a line segment
duplication of an angle

5. Given: $\angle x$ and $\angle y$, with $\angle x$ greater than $\angle y$, use construction instruments to:

double $\angle x$
bisect $\angle y$
construct an angle that would be equal to $2 \angle x - \frac{1}{2} \angle y$

6. In triangle ABC construct:

the median from A
the altitude from C

III.

PARALLEL LINES

LESSONS 1 and 2

Aim: to arrive at an acceptable definition of parallel lines
to discover the angle relationship propositions which are used to deduce that lines are parallel
to arrange these propositions in a miniature postulational system
to use the propositions in deducing that lines are parallel

Development:

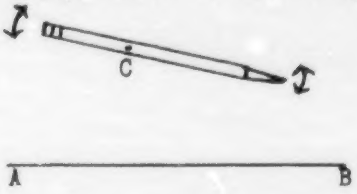
Definition of parallel lines: The student should realize that a new topic or sequence should begin with an acceptable definition for each new term used unless it is to be listed as "undefined." There are many illustrations of parallel lines (railroad tracks, etc.) in the world about us. Members of the class will therefore readily suggest possible definitions of parallel lines.

In establishing an acceptable definition for parallel lines, the frequent suggestion that they are always the "same distance apart" will provide an opportunity to discuss the requirements of a good definition. Since "distance" has neither been defined yet nor accepted as undefined, it should not be used in defining parallel lines. It may also be necessary for the teacher to challenge the class to include the requirement that the lines be "in the same plane" by exhibiting pencils or rulers held in a position to represent skew lines (lines not meeting which do not lie in the same plane). The definition of parallel lines should then be stated.

It should also be pointed out that segments of parallel lines are said to be parallel if they lie on parallel lines.

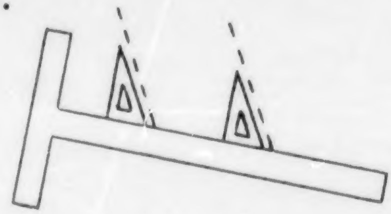
Experiment #1: How many lines are there parallel to line AB, and going through a point C outside of AB? Students should experiment by rotating a pencil through C until it attains a position in which it seems to be parallel to AB. They will usually arrive at the generalization:

(A) "There is one and only one line parallel to a given line through a given outside point." This should be accepted as a postulate. Note to teacher: The existence of at least one parallel line can be proven from the other axioms of Euclidean Geometry. Euclid's assumption was that there was not more than one.



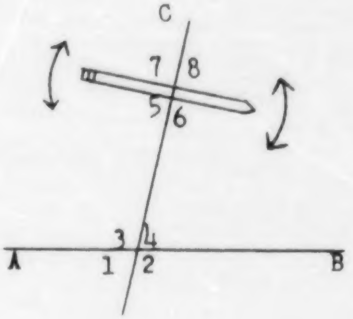
In accepting this generalization, the students should be lead to the question of how to construct this line. It will be readily seen that the definition of parallel lines is inadequate for this purpose.

Experiment #2: Students should be shown how a draftsman constructs parallel lines by the use of a triangle held in two different positions against a T-square. This will suggest the use of a reference line (represented by the T-square) cutting the two lines whose parallelism is being discussed.



Students should now be asked to continue the experiment in which a pencil is rotated, but with an added reference line (identified as a transversal) which joins C to any point of AB.

They should be asked to consider such questions as: Does any relation now in the figure help to identify the position in which the pencil is parallel to AB? (8 angles are formed and should be numbered.) Do any relationships appear to hold between some of these when the lines are parallel, but not to hold when they are not parallel?



In making the generalizations from these experiments, the need for the introduction of the terms exterior angles, interior angles, corresponding angles, alternate interior angles, and consecutive interior angles will arise naturally out of the students' efforts to describe certain angle pairs. The use of these terms implies the presence of the transversal.

Generalizations: Students will readily arrive at generalization (B) and (C) and, through teacher questioning, can be led to formulate (D):

(B) If the corresponding angles are equal, then the lines are parallel.
(C) If the alternate interior angles are equal, then the lines are parallel.
(D) If the consecutive interior angles are supplementary, then the lines are parallel.
(E) Two lines perpendicular to the same line are parallel.

The construction of a line parallel to a given line by equal alternate interior angles or equal corresponding angles (one may be done in class, one for homework) will now serve to verify and to reinforce the generalizations made above.

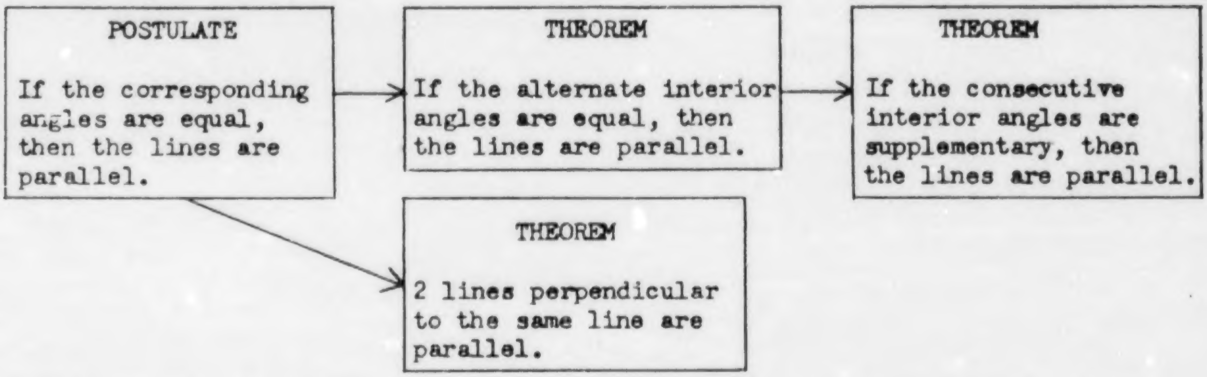
Students will frequently suggest accomplishing the construction by dropping a perpendicular from the given outside point to the given line and then erecting a perpendicular to this transversal at the outside point. This makes an excellent introduction to generalization (E). If this does not arise spontaneously, the teacher should attempt to get students to suggest this method and generalization.

Postulational System: The student should be led to realize that he has not yet given a deductive proof of any of the generalizations concerning parallel lines.

Can any of the generalizations be deduced from the others? Students may suggest that "2 lines perpendicular to the same line are parallel" is a special case of "equal corresponding angles." Discussion and informal proofs should further bring out that if we accept "equal corresponding angles" we can deduce "equal alternate interior angles" and that if we accept "equal alternate interior angles" we can deduce "equal corresponding angles." Does this mean that both are proved as theorems? Here is an opportunity to reinforce the concept of the invalidity of circular reasoning.

The student should be led to see that "supplementary consecutive interior angles" can either be a consequence of "equal alternate interior angles" or of "equal corresponding angles," or the latter two can be deduced from it. Therefore, if we choose to assume or postulate any one of these 3 generalizations, we can deduce the other two and also "lines perpendicular to the same line" (theorem) from it. The class should be allowed to make the selection and the generalizations should then be recorded under the appropriate "postulate" or "theorem" lists.

A diagrammatic representation of this miniature postulational system will help to promote understanding:



POSTULATE

If the corresponding angles are equal, then the lines are parallel.

THEOREM

If the alternate interior angles are equal, then the lines are parallel.

THEOREM

If the consecutive interior angles are supplementary, then the lines are parallel.

THEOREM

2 lines perpendicular to the same line are parallel.

The teacher should stress that the selection of postulates is arbitrary. This selection is subject to the desirability of keeping postulates to a minimum number necessary to deduce all the generalizations.

The Parallel Postulate: Generalizations (B), (C), (D), and (E) are in a sense dependent upon generalization (A) since (A) asserts the existence of the parallel lines for which the others show the sufficient conditions. (A) must therefore be a postulate and listed as such.

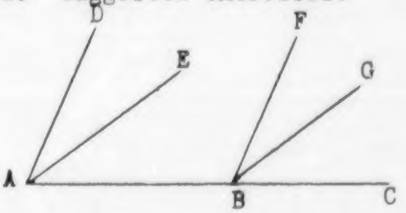
The teacher should now recount some of the history of the Parallel Postulate and point out that mathematicians, after failing in their efforts to deduce it from other postulates, examined the consequences of accepting some alternate postulates in its place (that there are 2 lines parallel to a given line through a point outside it or none parallel). The theorems deduced from each of these postulates form non-Euclidean geometries which are valid. Here is an opportunity to distinguish between truth and validity. Students will want to know which of the geometries is the "true" one. It should be pointed out that each of the geometries is useful in interpreting situations in the real world. The teacher may return to this point in discussing the sum of the angles of a triangle in a later lesson.

Ways to deduce lines parallel: What ways do we now know to deduce lines parallel? Propositions (B), (C), (D), and (E) should be recorded in notebooks under a new list headed "Ways to deduce lines parallel."

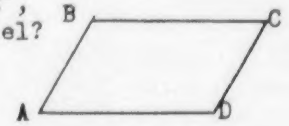
Deductive proofs involving parallel lines (see any textbook) should be done in class and for homework. In the more complicated diagrams, the necessity for a ready means of recognizing alternate interior and corresponding angles will arise. When pencil drawings are used, have the students outline in ink the sides of the angles. This will result in the characteristic "F" for corresponding angles and "Z" or "N" for alternate interior angles.

Numerical and algebraic exercises: It is suggested that the teacher introduce some numerical and algebraic exercises on the use of these propositions before the formal deductive exercises are attempted. Suggested exercises:

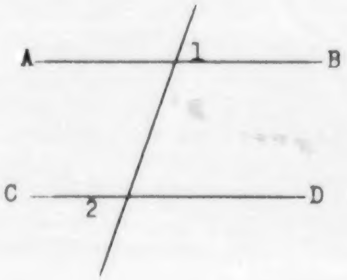
1. If $\angle DAB$ is 70°, and $\angle DAE$ is 30°, what angle would you need to know to prove that DA is parallel to FB? How many degrees would it have to contain? Answer the same questions for proving that AE is parallel to BG.



2. If $\angle A$ is x° and $\angle CDA$ is $180^\circ - x^\circ$, which lines can you prove parallel?



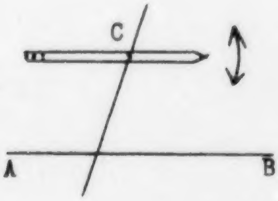
3. If $\angle 1$ is 80°, and $\angle 2$ is 80°, explain two ways in which you could prove that AB is parallel to CD.



LESSONS 3 and 4

Aim: to discover the properties of parallel lines
to apply these properties in deducing facts about angles
to introduce the nature of the converse

Development:



Experiment: In an experiment similar to the experiments in lessons 1 and 2 on parallel lines, students are instructed to move a pencil to a position such that the alternate interior angles appear equal, then to a position in which the corresponding angles appear equal, and finally to one in which the consecutive interior angles appear supplementary. In each case, what fact must result in regard to the position of the pencil in relation to AB?

Next, have students move the pencil until it appears to be parallel to AB. What facts must result in regard to the size of the alternate interior angles, the corresponding angles, and the consecutive interior angles?

Are the generalizations resulting from the second part of this experiment the same as those from the first? Consider: If you draw 2 right angles, they must be equal. If you draw 2 equal angles, must they be right angles? This is the first time that students have met the concept of converse. The definition of converse should now be given. Exhibit the hypothesis and the conclusion of one theorem written side by side with those of its converse. This arrangement will help to reinforce the meaning of converse.

One function of a converse is to suggest new propositions which may be formulated from the converses of accepted propositions. These converses may then be investigated. This procedure and the above experiment should lead students to state:

Generalizations:

(F) If two lines are parallel, then the corresponding angles are equal.
(G) If two lines are parallel, then the alternate interior angles are equal.
(H) If two lines are parallel, then the consecutive interior angles are supplementary.
(I) If two lines are parallel, then a line perpendicular to one of them is perpendicular to the other.

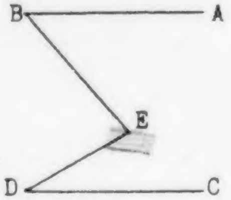
Generalizations (F), (G), (H), and (I) can now be arranged in a logical sequence (one selected as a postulate with the others deduced informally from it) as was done with the converses. Pupils should also list (F) and (G) under "Ways to deduce angles equal," (H) under "Ways to deduce angles supplementary," and (I) under "Ways to deduce lines perpendicular."

Deductive proofs of originals involving parallel lines (see any text) and these generalizations can now be undertaken. Some exercises should include cases in which lines are proved parallel by using angle relationships and in which angle relationships are also proved by using parallel lines. These will give students practice in distinguishing whether a theorem or its converse is the appropriate reason for a particular statement. Insistence on the "if-then" form of the statement of the parallel line generalizations will help avoid the error of using the converse instead of the correct reason (the "if" denotes the hypothesis of an already established deduction, the "then" indicates what we are deducing in this step).

Numerical and algebraic exercises: In addition to formal proofs, numerical and algebraic examples involving finding of angles connected with parallel lines are appropriate here.

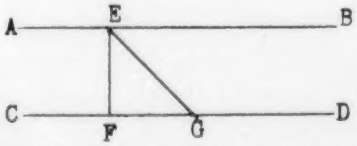
Suggested exercises:

1. If BA is parallel to DC, $\angle B$ is 50° and $\angle D$ is 30°, find $\angle BED$.



2. If 2 parallel lines are cut by a transversal, find the two interior angles on the same side of this transversal if one of them is 30° more than the other.

3. If AB is parallel to CD, $\angle EFD$ is $2x^\circ$, EG bisects $\angle FEB$, and $\angle EGD$ is $4x^\circ$, find x.



Converses: A more extensive treatment of the concept of the converse should begin with some non-geometric situations. The teacher can begin by writing two converse statements on the board: "If I am thirsty, I drink water" and "If I drink water, I am thirsty." Do these statements mean the same thing? Discussion of this and other situations, both geometric and non-geometric, should result in the conclusion that a true proposition may have a converse that is either true or false.

Students should be given exercises in forming the converses of statements, some of these converses being true and some being false. The students should be asked to judge the truth or falsity of the converse.

The danger of unsound reasoning by assuming the truth of a converse should then be discussed. We know that if an angle is an obtuse angle it is less than 180°; it is not true, however, that all angles less than 180° are obtuse. Exercises on identifying such unsound reasoning under the heading, "Assuming a converse" should be given to the class, using both geometric and non-geometric contexts.

Suggested exercises:

1. If two triangles are congruent, then a pair of corresponding altitudes are equal. Therefore, if two triangles have a pair of altitudes equal, they are congruent.

2. All good citizens vote. Since Mr. X voted, he must be a good citizen.

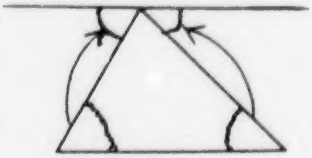
3. Advertising matter often provides a fertile source of illustrations of an appeal to unsound reasoning by "assuming a converse." A Fireball Motor ad reads, "If it's a Fireball Eight, it's a good car." This may be a perfectly sound statement, but the advertisers hope the public will assume that, "If it's a good car, it's a Fireball Eight."

The possibility of having either true or false converses to true propositions should be compared with the situation in definitions. Definitions are always reversible. A term and the words that constitute its definition are equivalent, and one may always be substituted for the other.

LESSONS 5, 6, and 7

Aim: to deduce the theorem on the sum of the angles of a triangle
to deduce a number of corollaries to the theorem
to apply these in deductive, algebraic, and numerical exercises

Development:



Experiment: Students are familiar with the fact that the sum of the angles of a triangle is 180°. The following experiment is suggested as a means of making the fact more vivid and of suggesting the procedure used in the deductive proof:

Have each student make a paper triangle, tear off the 2 base angles, and place them adjacent to the vertex angle as shown. He observes that a straight angle is formed.

Deductive proof: Have the students do a deductive proof of the theorem.

The nature of a postulational system: The teacher should now have the class trace back the deductive chain on which this theorem depends. If we were unwilling to accept the Euclidean Postulate, would the sum of the angles of a triangle be 180°? Tell the class that the sum of the angles can be deduced to be respectively less than 180° and more than 180° by using the 2 non-Euclidean postulates. Mathematicians once thought of measuring the angles of a huge triangle to determine which of the postulates was the "true" one, but found that no conclusion could be reached because the limitations of the measuring instruments produced results greater than, less than, and equal to 180° on different readings for the same triangle.

Corollaries: The corollaries to the "sum of the angles of a triangle" theorem can best be developed by having the students.

consider a numerical exercise (find the other acute angle of a right triangle in which one acute angle is 40°)

generalize from numerical exercises (the acute angles of a right triangle are complementary) deduce (informally) the generalization from "the sum of the angles of a triangle" theorem.

The following corollaries can each be treated in this manner:

- (a) A triangle can have at most one right angle or one obtuse angle.
- (b) The acute angles of a right triangle are complementary.
- (c) Each acute angle of an isosceles right triangle is 45°.
- (d) Each angle of an equilateral triangle is 60°.
- (e) The sum of the angles of a quadrilateral is 360°.
- (f) If two angles of one triangle equal two angles of another triangle, their third angles are equal.
- (g) The exterior angle of a triangle equals the sum of the two remote interior angles.

Item (g) above should be demonstrated both as a consequence of the angle sum theorem and directly from the parallel line theorems (by drawing a line through the vertex of the exterior angle parallel to the opposite side of the triangle).

Item (d) should be applied to the construction of the 60° angle and other angles which may be constructed from the 60° angle.

Numerical and algebraic exercises: The angle sum theorem and its corollaries should be applied in various numerical and algebraic examples.

Suggested exercises:

1. Find the number of degrees in each angle of a triangle whose angles are in the ratio 2:3:4.

2. The vertex angle of a triangle is 80°. The bisectors of its base angles meet at D. Find the number of degrees in the acute angle formed by the bisectors at D.

3. Angle A of triangle ABC is equal to x°. Represent, in terms of x, the number of degrees in the angle formed at the intersection of the two altitudes from vertices B and C. What have you proved about the relationship of this angle to angle A?

UNIT TEST

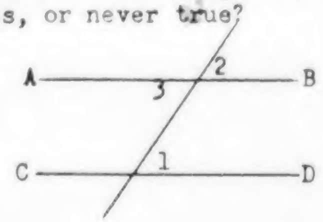
PARALLEL LINES

1. a) Through point P, a point outside a given line AB, construct a line parallel to AB.
   b) State the postulate or theorem which justifies the method you used in part a).

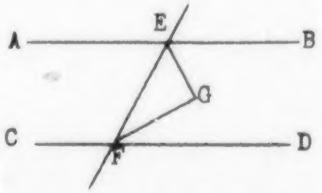
2. a) The exterior angle at the base of an isosceles triangle is 130°. Find the vertex angle.

   b) The exterior angles at two vertices of a triangle are acute angles. Is this statement always, sometimes, or never true?

   c) If AB is parallel to CD, $\angle 1$ equals $2x + 50^\circ$, and $\angle 2$ equals $4x - 10^\circ$, find angle 3.



   d) If AB is parallel to CD and the bisectors of $\angle FEB$ and $\angle EFD$ meet at G, find the number of degrees in angle G.



3. Tell whether each of the following examples of reasoning is sound or unsound. If unsound, explain why.

   In the tropics, it is very rainy in July. Therefore, if there is a very rainy week in the tropics, it must be July.

   In New York in December, the sun never sets later than 5:15 p.m. Therefore, if the sun sets at 5:03 p.m. one day in New York, it must be December.

4. Write the converse of: "If the median to the base of a triangle is perpendicular to the base, the triangle is isosceles."

5. Arrange the following statements in a logical sequence:

   The sum of the angles of a triangle is 180°.

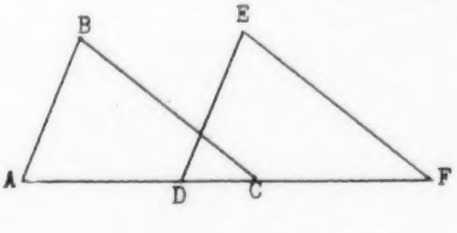
   Parallel lines are two lines in the same plane that do not meet.

   When two parallel lines are cut by a transversal, the alternate interior angles are equal.

   Through a given point, one and only one straight line can be drawn parallel to a given line.

6. Given: $AB \parallel DE$
   $AB = DE$
   $AD = CF$

   Prove: $BC \parallel EF$



7. Prove "The sum of the angles of a triangle is 180°."

IV.

CONGRUENT TRIANGLES CONTINUED AND TWO LOCUS THEOREMS

Lesson 1

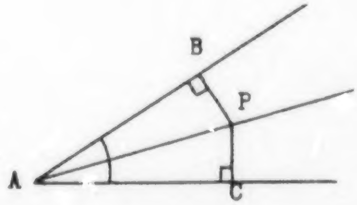
Aim: To define distance from a point to a line.
To develop another corollary of the "sum of the angles of a triangle" theorem, namely:
"Two triangles are congruent if two angles and a side opposite one of them are equal to the corresponding parts of the other" (SAA = SAA)
To deduce "Any point on the bisector of an angle is equidistant from the sides."

Development:

Ask a question to elicit what we should mean by the distance from the school to _____ Avenue. This leads to the postulate: The shortest line segment from a point, not on a line, to that line is the perpendicular from the point to the line. Therefore, we arrive at the definition: The distance from a point to a line is the length of the perpendicular from the point to the line.

Note: The above postulate will become an easily proven theorem if the first lessons on Inequalities are taught before these lessons.

Experiment: Have each student draw any angle and bisect it. From any point P on the angle bisector, drop perpendiculars to the arms of the angle. What appears to be true from the diagram? Can we deduce it?



The class can be led to deduce the generalization "any point on the bisector of an angle is equidistant from the sides." The deduction would probably be done by using "if two angles of one triangle equal two angles of another triangle, the third angles are equal," and this in turn leads to the ASA = ASA congruence principle.

Have the students realize that the two triangles agreed in a side and two angles which did not include that side. However, these triangles were shown to be congruent. Will such triangles always be congruent under these conditions? Have students deduce "If a triangle has a side and two angles which do not include this side equal to the corresponding parts of another triangle, the triangles are congruent."

Teacher: Shall we list this in the same list in which we placed ASA = ASA, SSS = SSS, and SAS = SAS?

Student: No. We postulated those three methods after experiments. This method we deduced. Therefore, it is a theorem and should be listed as such. (Have members of the class list the SAA = SAA theorem.)

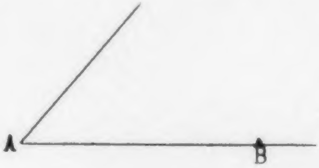
Have the students now state the generalization from the experiment: "Any point on the bisector of an angle is equidistant from the sides." This should now be deduced informally using SAA = SAA and listed as a theorem.

LESSONS 2 and 3

Aim: to deduce the converse of "base angles of an isosceles triangle" theorem

Development:

Experiment: At point B construct an angle equal to ∠A so that the angles will be consecutive interior angles. What do you notice?

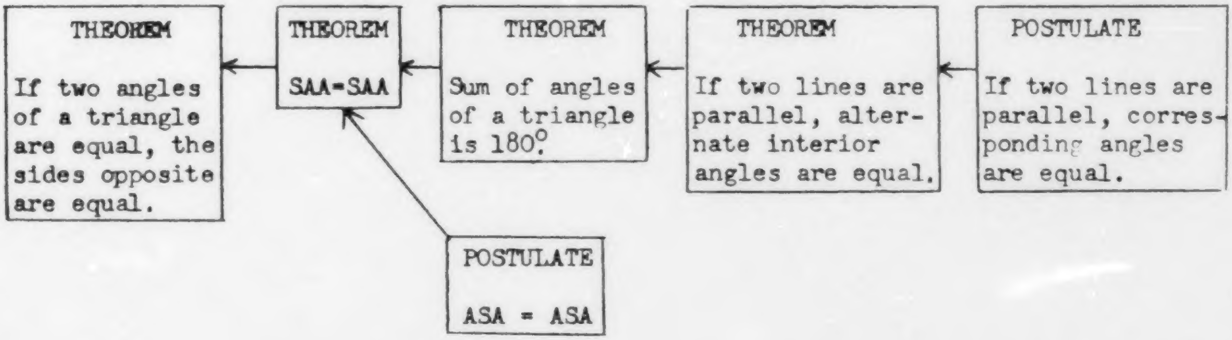


The students will discover the isosceles triangle and can be led to realize that it appears reasonable: "If two angles of a triangle are equal, the sides opposite these angles are equal." They should recognize that this is the converse of the "base angles of an isosceles triangle" theorem.

Teacher: We deduced that if two sides of a triangle are equal, the angles opposite these sides are equal. Can we list the converse now as a theorem?
Student: No. We must deduce it too. The converse of a theorem is not necessarily valid.
Teacher: Why didn't we deduce the converse when we first deduced the theorem?
Student: The deduction of the converse probably depends upon some new theorem.

Have students deduce "If two angles of a triangle are equal, the sides opposite these angles are equal." They will probably suggest drawing in the angle bisector as was done in the original theorem. The triangles are easily shown congruent by SAA = SAA. The converse can now be listed as a theorem.

Have the students discuss the sequence:


THEOREM
If two angles of a triangle are equal, the sides opposite are equal.
THEOREM
SAA=SAA
THEOREM
Sum of angles of a triangle is 180°
THEOREM
If two lines are parallel, alternate interior angles are equal.
POSTULATE
If two lines are parallel, corresponding angles are equal.
POSTULATE
ASA = ASA

Teacher: This converse makes us more powerful in geometry. Why?
Student: It gives us a new method of deducing that lines are equal.

Have the students add the converse to "Ways to deduce lines are equal."
If the lines are in the same triangle, deduce the angles opposite these sides are equal.

Apply the converse to original exercises (see any text).

LESSONS 4 and 5

Aim: to discover that SSA = SSA does not necessarily make two triangles congruent

to deduce "two right triangles are congruent if the hypotenuse and a leg of one are equal to the corresponding parts of the other"
to deduce the converse of "any point on the bisector of an angle is equidistant from the sides"

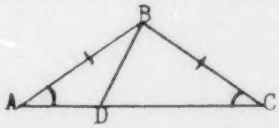
Development:

Teacher:	We have found that triangles are congruent if SAS = SAS, or SSS = SSS, or ASA = ASA, or SAA = SAA. What question do you think a mathematician would naturally ask next?
Student:	Are there other ways to show triangles congruent?
Teacher:	Can you suggest some other possibilities which we might examine?
Student:	Perhaps AAA = AAA, or SSA = SSA might be such other ways.
Teacher:	We call this method of discovering new propositions "Reasoning by Analogy." The propositions suggested for examination have elements which are similar to the propositions already accepted or deduced. You should recall earlier in the term we examined such statements: Joan is a very good student in mathematics. Therefore, she will be a very good student in social studies.

Have the students realize that analogy is a good way to set up or discover a new proposition for investigation, but it is not a valid way of arriving at the "truth" of the proposition.

Experiment: Have the students examine the possibility of AAA = AAA as a method of showing triangles congruent. They should be led to see that this is an incorrect way of deducing congruence. Students discovered that triangles having three angles of one equal to three angles of the other are said to be similar in Ninth Year Mathematics.

Experiment: Have the students examine the possibility of SSA = SSA as a method of showing triangles congruent. Draw any isosceles triangle. Draw any line from the vertex to the base. Triangle ABD and triangle BDC have SSA = SSA. Are they congruent?

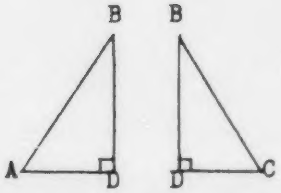


The teacher might cut out triangle ABD and BDC and point out that they agree in SSA. However, when the students attempt to make the triangles coincide, matching CB and AB, ∠C and ∠A, they find the triangles do not coincide and are not congruent.

Have the students examine their original drawing and consider under what conditions the triangles would be congruent. The conditions would be that

BD is a median (SSS = SSS)
BD is an angle-bisector (SAS = SAS)
BD is an altitude (SAA = SAA)

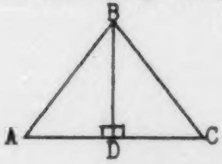
Have the students consider the case when BD is an altitude. Cut out two triangles as shown in the diagram at right.



Teacher: I now have two separate triangles which agree in these lines (hypotenuses), in these lines (BD), and in these right angles. Are the triangles congruent?

Student: They have SSA = SSA. They are not necessarily congruent.

Teacher: What happens when I place the triangles together in this manner?



Student: If we put the triangles together, they form an isosceles triangle whose base angles are equal. Therefore, they are congruent because SAA = SAA.

Teacher: Whenever we have two separate triangles, can we put them together by matching their equal sides and in this way form a big triangle? (Show two triangles with a pair of equal sides matched which does not form a triangle. For example, match the hypotenuses of the two right triangles, or match AB and BC of the original two triangles.)

Student: No. They usually form a four-sided figure.

Teacher: Why did the two triangles form a large triangle earlier? (Show again.)

The students will agree that the right angles made it possible to form a large triangle from the two triangles.

Have the students now state the proposition: "Two right triangles are congruent if the hypotenuse and a leg of one are equal to the corresponding parts of the other." (The students may say that the triangles agree in side, side, and right angle. The teacher should agree, but introduce the conventional terminology.)

Use the models and have the students do the deduction orally.

Give careful attention to the three steps which show that the large figure is a triangle and not a quadrilateral. Have students do the deduction and place theorem in list.

Apply the theorem to exercises.

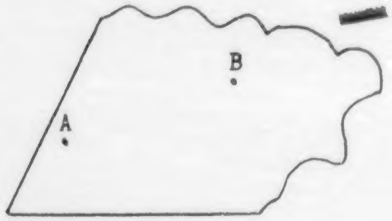
Have the students state the converse of "Any point on the bisector of an angle is equidistant from the sides." Deduce this converse and place in list.

LESSON 6

Aim: to introduce the idea of locus
Note: The idea of locus should be developed through the year particularly in relation to coordinate geometry

Development:

Have the students consider the following challenge:



> A treasure is buried on an island. Two hints are given. The treasure is known to be (1) equidistant from the two straight shore lines and (2) it is equidistant from the two palm trees A and B. Where is the treasure?

The first hint will suggest an angle-bisector. Have the students draw the angle-bisector of the two straight shore lines and color it.

Teacher: How did you know that all of these points were equidistant from the straight shore lines?

Student: We have already deduced that "Any point on the angle-bisector is equidistant from the sides of the angle."

Teacher: Perhaps some other points like this one (point to one not on the angle-bisector) is also a point equidistant from the sides of the angle.

Student: If that point were equidistant from the sides of the angle, it would lie on the angle-bisector. We have already deduced that too.

The teacher should point out that it is necessary to establish both a theorem and its converse before we can be sure that we have located the set of points which satisfy a condition. The notion of a set implies that it contains all the elements, and only those elements, which satisfy a given property.

Teacher: We call this colored line the locus (or set) of all points which satisfy the first hint. Who can suggest a locus of all points which satisfy the second hint?

Student: A point halfway between A and B.

Teacher: Is that the only point which satisfies the hint "equidistant from A and B?"

Students will quickly see that all the points of the perpendicular bisector satisfy the condition. They will be eager to note that the one point common to each locus is the place to dig.

The teacher should now formalize the discussion. Suggest the following format for locus theorem: The locus of all points which ______ is ______. The blanks can be filled in only when both (1) a theorem and (2) its converse have been established.

The two statements relating to "equidistant from 2 lines" have been given.

> Any point on the angle bisector is equidistant from the sides.
> Any point equidistant from the sides of the angle is on the angle bisector.

Have the students state and deduce orally the two statements related to "equidistant from two points."

> If a point is equidistant from two points, it is on the perpendicular bisector of the line segment joining the two points.
>
> If a point is on the perpendicular bisector of a line segment, it is equidistant from the ends.

Have the students list as a theorem the one locus statement noted as a combination of the two angle bisector statements.

> The locus of points within an angle and equidistant from the sides of the angle is the bisector of that angle.

Have the students list as a theorem the one locus statement noted as a combination of the two perpendicular bisector statements.

> The locus of points equidistant from two given points is the perpendicular bisector of the line segment joining the two points.

Apply the theorem to exercises involving each, or both of the loci theorems.

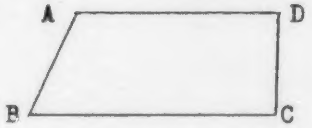
UNIT TEST

CONGRUENT TRIANGLES CONTINUED AND TWO LOCUS THEOREMS

1. Write in order the sequence of propositions leading to the SAA = SAA theorem.

2. Deduce: If two angles of a triangle are equal, the sides opposite are equal.

3. a) Find by construction a point on BC which is equidistant from points A and B.



   b) State the locus theorem which justifies your construction.

4. a) Deduce: "Any point equidistant from the sides of an angle lies on the bisector of that angle."

   b) State the converse of the theorem in a).

   c) Why must we deduce both theorem and converse before being satisfied that the set of points on the angle bisector contains all the elements which are equidistant from the sides of the angle and no others?

V.

QUADRILATERALS

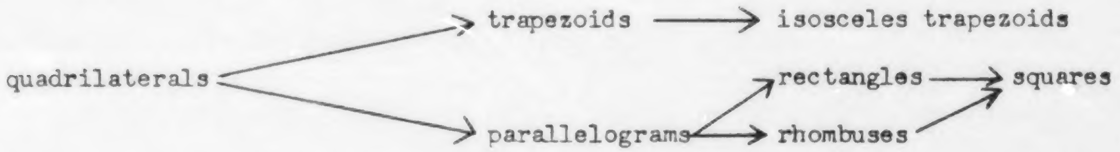
LESSON 1

Aim: to define quadrilaterals

Development:

Have the students review that quadrilaterals may assume the special forms of square, rectangle, and parallelogram. To these should be added the rhombus, trapezoid, and isosceles trapezoid.

Have the students list the quadrilaterals and consider a sequence starting with the quadrilateral which is itself a subset of polygons. They should use the concepts of superset and subsets in defining the quadrilaterals. Subsets of quadrilaterals are parallelograms and trapezoids. As each of these is developed into their subsets, the following pattern will evolve:



The project of formal definitions may now be undertaken. It should be clear that the classifying parts of the definitions have been determined. Thus, "A rectangle is a parallelogram which..., or "An isosceles trapezoid is a trapezoid which...." It remains to determine the distinguishing properties which permit us to distinguish the object being defined from other members of its superset.

The definitions for trapezoids and isosceles trapezoids should be undertaken first since they present no serious difficulties and should be recorded in notebooks as definitions.

The teacher will find that some students will suggest as the distinguishing property of parallelograms that the opposite sides are parallel, others that the opposite sides are equal, and some that the opposite sides are both equal and parallel. The students may suggest the following about the opposite sides of a quadrilateral:

whenever they are parallel, they are also equal
whenever they are equal, they are also parallel

Have the students conduct experiments to show that both statements are true. These should be done quickly. The results of the experiments suggest that only one of the properties is needed for our definition. Mathematicians have agreed to use the parallelism rather than the equality conditions. For this reason it is called a parallelogram rather than an equalogram.

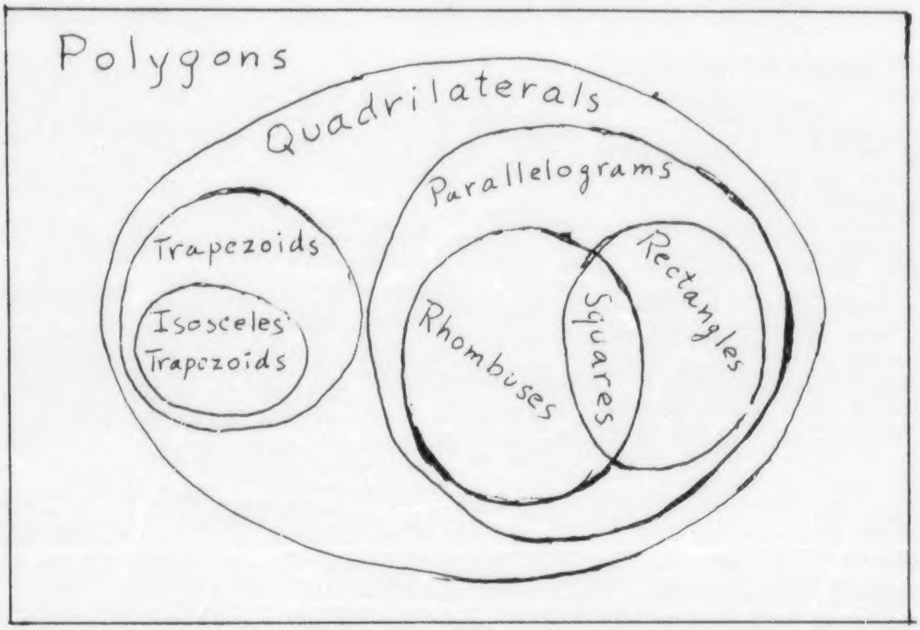
The definition of a rhombus should be "a parallelogram in which two adjacent sides are equal." It is natural that this will arouse the students' curiosity concerning the equality of all the sides. A number of students will offer deductive proofs that this is so. Do not discourage these offers. If none are offered, an experiment should be used to examine the question.

Note: In this lesson, we are suggesting the philosophy of definitions which prefers that the distinguishing properties be reduced to a minimum. This ideal unifies the subject matter for young students but the teacher should recognize that, since definitions are arbitrary, there is nothing inherently wrong with the definition: A rhombus is a quadrilateral with four equal sides, or even A square is a quadrilateral with four equal sides and four equal angles. However, another advantage of the minimal property type of definition is that it requires one to deduce fewer facts in order to establish that a specific figure is a _____.

The rectangle may be defined as "a parallelogram having one right angle." The students' curiosity of whether the three other angles must also be right angles may be satisfied by an experiment or a deduction. In most cases the student will offer the deduction.

The square may now be defined either as "a rhombus having one right angle" or as "a rectangle having two adjacent sides equal." The square is thus identified as a set of parallelograms which are members of the set of rectangles and the set of rhombuses. If the interest and ability of the class warrants, the teacher may introduce the concept of the intersection of two sets. It would follow that the set of squares is the intersection of the set of rectangles and the set of rhombuses.

Venn Diagrams are advantageous for summarizing this lesson as follows:


Polygons
Quadrilaterals
Parallelograms
Trapezoids
Isosceles Trapezoids
Rectangles
Rhombuses
Squares

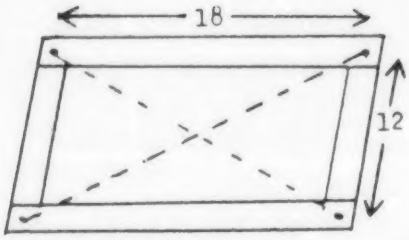
All these definitions should be recorded by students in their list of definitions.

LESSONS 2 and 3

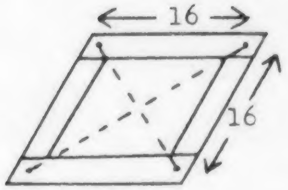
Aim: to explore and discover by experimentation properties of quadrilaterals and subsets of quadrilaterals
to use these discoveries in numerical problems using arithmetic and algebra

Development:

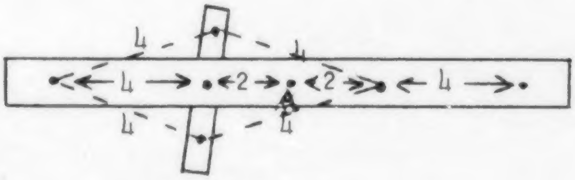
Have the students suggest the properties of opposite sides, opposite angles, consecutive angles, and diagonals of a parallelogram. To examine these suggestions, experiments should be performed quickly and the results noted informally. To facilitate these and other experiments, the teacher may avail herself of one or more of three models made of wood or metal strips. They are hinged as indicated in the following figures: (the indicated dimensions are suggested as best fitted for visibility in demonstration)



Model 1



Model 2



Model 3

Each model is capable of assuming various shapes. Model 1 may assume the shape of a rectangle as well as a non-rectangular parallelogram. Model 2 may assume the shape of a square as well as non-square rhombuses. Model 3, with the slats acting as diagonals, may assume shape of kite, or parallelogram.

Note: Teachers have reported that these models may be easily made with the metal strips found in Erector Sets. Lengths and angles are easily measured.

Have the students extend the experimental program to include quadrilaterals and trapezoids. The results may be summarized in table form as follows:

QUADRILATERALS

Four sides (definition)
Sum of angles is 360°

Trapezoid

One pair of parallel sides (definition)

Isosceles Trapezoid

Non-parallel sides are equal (definition)
Base angles are equal

Parallelogram

Opposite sides parallel (definition)
Opposite sides equal
Diagonals bisect each other
One diagonal divides it into two congruent triangles
Opposite angles equal
Consecutive angles supplementary

Rhombus

Two adjacent sides equal (definition), it is equilateral
Diagonals perpendicular
Diagonals bisect the angles
Diagonals divide it into four equal right triangles

Rectangle

One right angle (definition)
All angles are equal
Diagonals are equal

Square

The students should understand that the properties of any subset include those of its supersets.

Have the students consider numerical exercises whose solutions depend upon knowledge of one or more of the properties of the quadrilaterals.

Suggested exercises:

1. One angle of a parallelogram is 50°. Find the other three angles.
2. ABCD is a rhombus and $\angle A = 50^{\circ}$. Find the number of degrees in $\angle ABD$.
3. In rectangle ABCD, diagonal $AC = 5x + 5$ and diagonal $BD = 7x - 7$. Find x and the length of each diagonal.
4. In isosceles trapezoid ABCD, the base $\angle A = 50^{\circ}$. Find the other angles.
5. ABCD is an isosceles trapezoid. $\angle A = 5x - 20$ and $\angle C = 3x + 40$. Find x and all the angles of the trapezoid.

Lesson 4

Aim: To select the more important of the propositions discovered as true in the previous lesson and to elevate them to the rank of theorem.

Note: They will achieve this rank not only because they will be proved by deduction, but because they are going to be useful in subsequent deductions.

Development:

Prove quickly and informally all the properties of a parallelogram as discovered in Lessons 2 and 3.

1. If a quadrilateral is a parallelogram, a diagonal divides it into two congruent triangles.
2. If a quadrilaterial is a parallelogram, its opposite sides are equal.
3. If a quadrilateral is a parallelogram, its diagonals bisect each other.
4. If a quadrilateral is a parallelogram, its opposite angles are equal.
5. If a quadrilateral is a parallelogram, its consecutive angles are supplementary.

This lesson affords an opportunity to reiterate the two values of deduction as spelled out in the Introduction to this guide. The teacher must be sure the students continue to distinguish between the inductive approach of Lessons 2 and 3 and the rearrangement into a postulational sequence. These 5 theorems do not necessarily form a consecutive sequence but interesting arrangements of them can be made.

Also, because of the multiplicity of theorems to be quickly deduced, this lesson provides an opportunity to use socialized recitation and committee procedures.

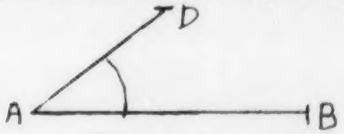
Students should now add to their "Methods of deducing" lists new methods of proving line segments equal, and angles equal.

Lesson 5

Aim: To study the converse theorems which furnish sufficient conditions to prove that a quadrilateral is a parallelogram.

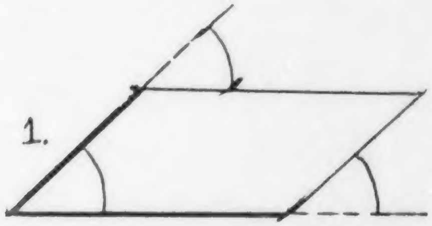
Development:

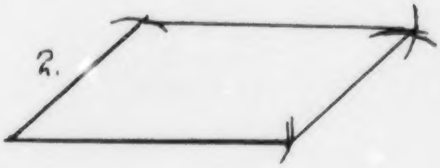
Challenge: Given:

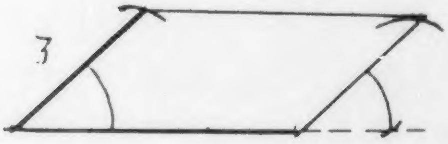


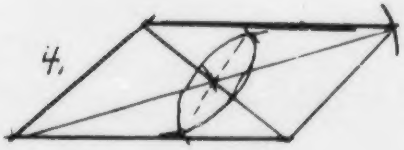
Construct: parallelogram ABCD

As students try this construction, circulate about the room and place on board samples of the following four approaches:









As each student explains his method, elicit and list the assumptions they made:

1. If the opposite sides of a quadrilateral are parallel, it is a parallelogram.
2. If the opposite sides of a quadrilateral are equal, it is a parallelogram.
3. If one pair of sides of a quadrilateral are equal and parallel, it is a parallelogram.
4. If the diagonals of a quadrilateral bisect each other, it is a parallelogram.

Teacher: May we assume all these facts?
Student: The first one is valid because it is the converse of the definition and is automatically true.
Student: The others are converses of theorems we proved yesterday and must be proved.

Informal proofs (committee work is possible) will prove all three. List as theorems and make a list, "Methods of deducing a quadrilateral is a parallelogram" with 4 methods:

1. Show both pairs of sides $\parallel$
2. Show both pairs of sides =.
3. Show one pair = and $\parallel$
4. Show diagonals bisect each other.

Consider converses of other properties of parallelograms.
   a. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram (true).
   b. If a diagonal of a quadrilateral divides the figure into two congruent triangles, the figure is a parallelogram (false).

Note: This lesson can also be accomplished by laying sticks on the overhead projector and forcing a parallelogram in the various ways.

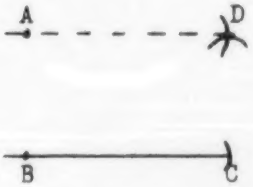
LESSONS 6, 7 and 8

Aim: to provide practice with deduction exercises related to the parallelogram theorems
to develop further methods of proving that quadrilaterals are rectangles, rhombuses or squares

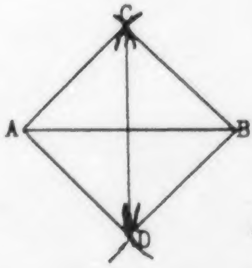
In addition to the traditional deduction exercises found in textbooks, the teacher should include some constructions which are related to the theorems.

Suggested exercises:

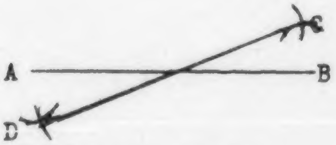
1. By making AD=BC and DC=AB, we can prove that AD $\parallel$ BC. This provides an alternate method of constructing a line through a given point parallel to a given line.



2. By making AC=BC=AD=BD, we can prove that CD is the perpendicular bisector of AB. What parallelogram property permits this conclusion?



3. By making AC=BD and AD=BC, we can show that DC bisects AB. How?



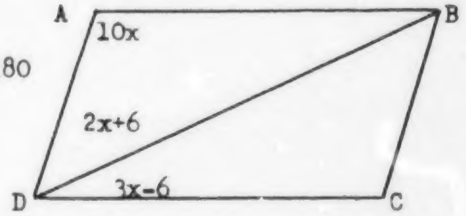
Methods of constructing a rhombus, rectangle or square may be related to methods of proving that a quadrilateral is a rhombus, rectangle or square respectively.

Have the students study methods of proving that quadrilaterals are rectangles, rhombuses or squares. The emphasis should be placed upon their respective definitions as the method most often used. Other methods deserve mention and should be viewed as deduction exercises. Students should list these as new "How to deduce" methods. Some of these other methods are:

If the diagonals of a parallelogram are equal, the figure is a rectangle.
If a diagonal of a parallelogram bisects one of its angles, the figure is a rhombus.
If the diagonals of a parallelogram are equal and perpendicular to each other, the figure is a square.

Suggested exercises:

1. Given ABCD is a parallelogram



   a. Write an equation in x.

   Answer: $10x + 2x + 6 + 3x - 6 = 180$

   b. Solve for x.

   Answer: $x = 12$

   c. Find the three indicated angles.

   Answer: $\angle A = 120^\circ$, $\angle ADB = 30^\circ$, $\angle BDC = 30^\circ$.

2. Given ABCD is a parallelogram with diagonals intersecting at E.
   $AE = 3x - 3y$, $EB = 2x + 5y$, $EC = 12$, $BD = 30$

   a. Form two equations in x and y.

   Answer: $3x - 3y = 12$
   $2x + 5y = 15$

   b. Solve for x and y.

   Answer: $x = 5$; $y = 1$

   c. Is parallelogram ABCD a rectangle? Why?

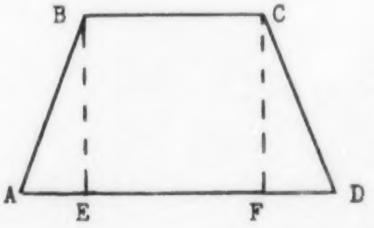
   Answer: No. It is not a rectangle because the diagonals are not equal.

Lesson 9

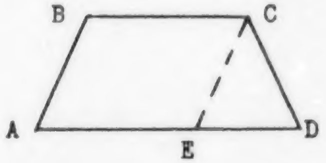
Aim: To deduce informally that the base angles of an isosceles trapezoid are equal.
To deduce informally that the diagonals of an isosceles trapezoid are equal.

Development:

Have the students consider two methods of deducing that the base angles of an isosceles trapezoid are equal. In one method perpendiculars are dropped from the ends of the smaller base onto the larger. Thus, a rectangle and two congruent triangles are produced. (This method is valuable in solving some area problems involving isosceles trapezoids.)



In the second method a line is drawn through an end point of the smaller base parallel to the remote leg as shown in the diagram at the right. Thus, a parallelogram and a triangle, which can be proved isosceles, are formed.



Have the students deduce that the diagonals of an isosceles trapezoid are equal. The converse of this proposition is suitable for an honor or extra-credit problem.

Exercises may be found in any textbook.

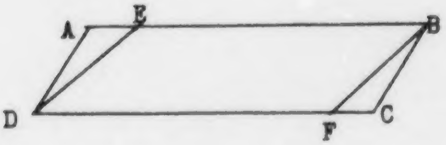
UNIT TEST

QUADRILATERALS

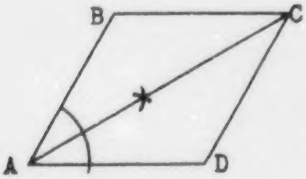
1. From the terms quadrilateral, parallelogram, rectangle, rhombus and square, choose the largest set for which it is true that
    a. The diagonals bisect each other.
    b. The sum of its angles is 360°.
    c. The diagonals bisect its angles.
    d. It is equiangular.
    e. The consecutive angles are supplementary.
2. Deduce: If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.
3. ABCD is a parallelogram. What minimum additional data would you need to deduce that:
    a. it is a rectangle
    b. a rhombus
    c. a square

4. Given: ABCD is a parallelogram
AE = CF

Deduce: DEBF is a parallelogram.



5. In this construction $\angle A$ was bisected, BC was drawn $\parallel$ to AD, and CD was then drawn $\parallel$ to BA. Prove that ABCD is a rhombus.



Note 1: One possible sequence is to teach Area right after Quadrilaterals. The Area sequence in this spot enables one to avoid the difficult concepts of incommensurability in the postulate of the Similar Triangle Sequence.

Note 2: Many teachers prefer to introduce the topic of coordinate geometry here and use it, for contrast and simplification, wherever feasible for the rest of the year. Others prefer to postpone this unit until after the Pythagorean Theorem, since that makes the correspondence with real numbers more meaningful, as far as the irrationals are concerned. The poorer students enjoy this unit and there is an advantage in introducing it here. Teachers who wish to postpone it will easily be able to use these lessons in combination with the others which appear on pages 73, 90-92, 100-105, 110-113, 120.

VI.

INTRODUCTION TO COORDINATE GEOMETRY - AN EXAMPLE OF ANOTHER POSTULATIONAL SYSTEM

Lesson 1

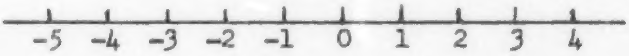
Aim: To introduce the fundamental postulate of coordinate geometry.
To introduce basic definitions and theorems of this postulational system.

Development:

The fact that point is undefined has probably bothered students. Tell them about the effort of Descartes to locate a point in a plane by setting up a 1:1 correspondence between pairs of real numbers and points in the plane. Discuss the revolution in mathematics in the Seventeenth Century due to this marriage of algebra and geometry.

Postulate: Every straight line may be considered as an infinite set of points which can be put into 1:1 correspondence with the set of real numbers. This implies that we can select any point on the line and assign the number zero to that point. This point is called the Origin, O. We then select any other point, P, and assign to this point the number 1. Thus OP becomes the unit distance. It could be 1 inch, 1 centimeter, 1 micron, 1 mile, 1 light year, etc. The unit on the blackboard will always be different from that in a notebook. Now, points A, B, C, D, etc. with the coordinates 3.6, 159, -2, $\sqrt{5}$ etc. can readily be given positions on the line.

Recall the number line of Ninth Year Mathematics.



and have students indicate the position of points M (3.6), N($\sqrt{5}$), R(-2), etc. The irrationality (incommensurability) of $\sqrt{5}$ will be clarified later in the year.

Thus, if we refer to a point K as having coordinate 'a', we mean the measure of distance OK is the absolute value of the real number 'a'. Students will need to be shown that 'a' can be a negative number. We use the symbol $|OK|$ to represent the distance from O to K. Thus, if $a = -5$, $|OK| = 5$.

Now show that the Cartesian Coordinate System, using two real number lines, perpendicular to each other at their common origin, associates every point in the plane with a pair of real numbers. Show how (-2,3) and (3,-2) locate different points so must be different number pairs, and thus that the number pairs are 'ordered pairs.'

Postulate: There is a 1:1 correspondence between the set of points in a plane and the set of ordered pairs of real numbers.

Have students review the definitions of axes, x axis, y axis, coordinates, ordinate, abscissa.

Note that in this separate postulational system, we will accept all our postulates and theorems of Euclidean Geometry plus the two new postulates.

A simple theorem of coordinate geometry is that in the coordinate grid, all lines parallel to the Y axis (X axis) are perpendicular to the X axis (Y axis) and are equally spaced.

Also prove (informally) the theorems:

Two points with the same abscissa (ordinate) lie on a line parallel to the Y axis (X axis).

The distance between two points with the same abscissa (ordinate) is equal to the absolute value of the difference of their ordinates (abscissas). Represent this distance by $|y_2-y_1|$ or Δy ($|x_2-x_1|$ or Δx)

Lessons 2 and 3

Aim: To develop a method of finding the midpoint of a line segment if the segment is parallel to either axis.
To apply coordinate geometry to exercises with specific points.

Development:

First, note a set of points like (5,7), (2,7), (-1,7), (8,7) etc. and evolve the idea of the equation of a line.

The set of points (x,c) where x is any real number and c is a constant may be represented by the equation $y = c$.

The set of points (c,y) where y is any real number and c is a constant may be represented by the equation $x = c$

If a line segment is parallel to the Y-Axis (X-Axis), the ordinate (abscissa) of its midpoint may be found by averaging the ordinates (abscissas) of its endpoints

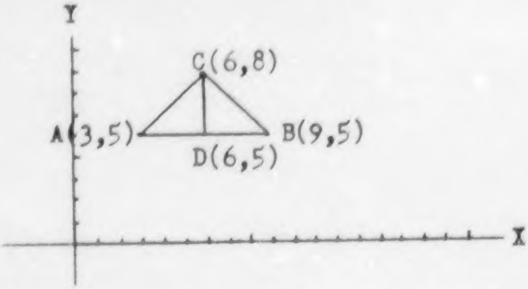
$$\frac{y_1 + y_2}{2} \qquad \left(\frac{x_1 + x_2}{2}\right).$$

Suggested exercises:

1. Plot (3,5), (9,5), and (6,8). What kind of a triangle is formed?

At this stage, students cannot show that the triangle is isosceles by finding the lengths of its sides because they know only the lengths of line segments on a line parallel to an axis.

What other ways have we of showing that a triangle is isosceles?



Students can write the explanation in paragraph form which is as essential to their training as the demonstrative proof form. They need to quote reasons only when the reasons are not one of the theorems and definitions of this unit.

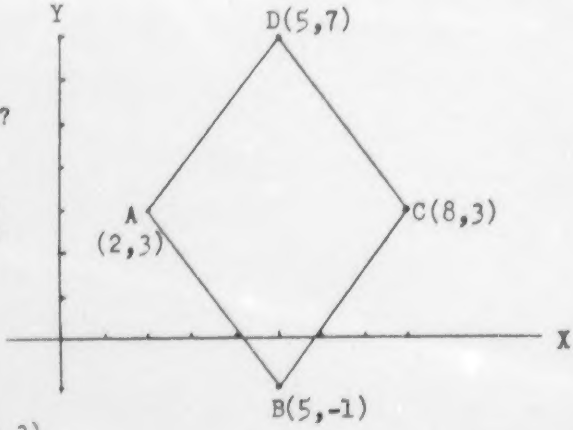
Acceptable student answer:

Line CD ⊥ AB and since D is (6,5), it is the midpoint of AB. Therefore, the triangle is isosceles because if the altitude to the base of a triangle bisects it, the triangle is isosceles. (Another reason for the triangle being isosceles is any point on the perpendicular bisector of a line segment is equidistant from the ends of the line segment.)

2. Plot A(2,3), B(5,-1), C(8,3), D(5,7)

a. Show ABCD is a parallelogram.
b. Why isn't it a rectangle?
c. What special parallelogram is it? Why?

Student thinks of the four ways to prove a figure is a parallelogram. Because the sides do not lie on lines parallel to the axes, he chooses to try to show that the diagonals bisect each other.



Acceptable student answer:

a. AC || X-Axis, and so its midpoint is (5,3).
BD || Y-Axis, and so its midpoint is (5,3).
Therefore, ABCD is a parallelogram because the diagonals have the same midpoint, and "if the diagonals of a quadrilateral bisect each other, it is a parallelogram."

Note: To obtain the midpoint of BD, the student must average 7 and -1. This helps to reinforce addition of signed numbers. Bright students can also explore another way of getting the average of two numbers by splitting their difference.

b. AC is 6, DB is 8. Hence, the figure is not a rectangle because the diagonals are not equal.

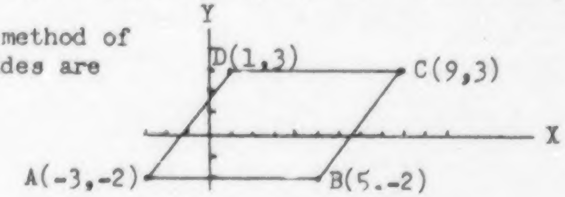
Note: This is a good place to teach the meaning of contrapositive.

c. AC and DB are perpendicular to each other. Therefore, ABCD is a rhombus because 'if the diagonals of a parallelogram are perpendicular, it is a rhombus.'

Note: If your class keeps a growing list of the theorems which it has proved, the theorem used in c may well have been one of them, even though it is not listed as one of the theorems in the syllabus.

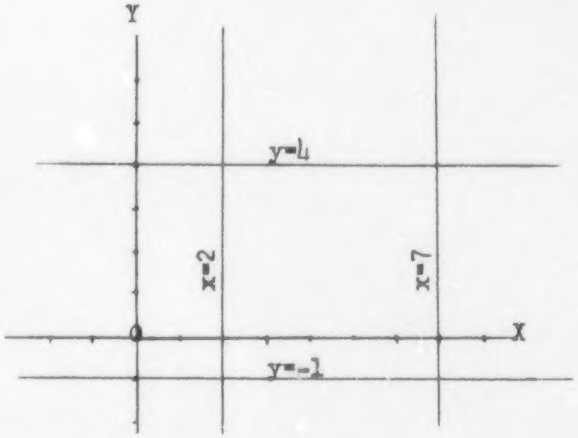
3. Plot A(-3, -2), B(5, -2), C(9,3) D(1,3). Why is ABCD a parallelogram?

The student may select the method of showing that one pair of sides are equal and parallel.



4. Draw x = 2, x = 7, y = 4, y = -1. Why is the closed figure formed a parallelogram? Why is it a rectangle?

Note: These lines may be called 'the locus of points which satisfy the equations x = 2, x = 7, y = 4, and y = -1,' provided that the word "locus" has been introduced earlier in the year. The first two locus theorems can be taught directly after "hypotenuse arm" theorem, thus giving students almost a full year's exposure to this difficult concept. It is reinforced here for the first of many times during the year. Or, the concept of locus may be taught for the first time in this unit of work.



5. Write the equation of a line whose abscissa is always equal to its ordinate.

Student response: $x = y$

The teacher should have the students plot several points to be sure they realize that the line contains infinitely many such points. These points satisfy the equation $x = y$, and conversely, if two numbers x and y corresponding to a point satisfy the equation $x = y$, the point lies on this line. The equation of the line is therefore an algebraic representation of the locus of points whose abscissas are equal to their ordinates.

In subsequent assignments, continue to develop this idea with examples such as: "Write the equation of a line whose ordinate is always two less than its abscissa."

This exercise could be postponed until the topic of locus is completed.

> Note: Later in the year we will use general coordinates such as (a,b) and use the tools of coordinate geometry to prove theorems instead of merely to do exercises such as 1-5 above.

VII.

CIRCLES

Note: This unit may be postponed until after similarity and areas, in which case the teacher will wish to enrich the unit with many numerical exercises, and originals involving similarity in circles.

Lesson 1

Aim: to reinforce the understanding of the necessity of definitions and postulates in building up a deductive sequence
to make evident the physical importance of circles in our civilization
to develop the idea of arc degrees
to reach agreement on two postulates

Development:

Have the students note the great strides made by mankind after he invented the wheel. Discuss the importance of circles in civilization through their use in gears, pistons, cylinders, and so on.

The students should state that they must decide upon a definition of what is a circle. Have them use their knowledge of what they do with compasses in drawing diagrams of circles to arrive at the definition:

A circle is the set of all points whose distance from a fixed point is constant.

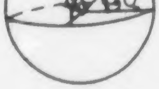
It is necessary to inventory such familiar and previously defined terms as radius, central angle, arc, and diameter. Insist upon the form which places each in its superset. For example,

An arc is a curved line which is part of a circle.

Experiment: Draw two unequal circles. Put a 30° central angle in each with a protractor. (The purpose of this exercise is to get a tactile familiarity with the relationship between central angles and their intercepted arcs.) The drawings will reveal clearly that the arcs which are intercepted are not equal. (Note that by equal we mean congruent.)

To reinforce this difference in lengths of arcs, draw a sphere to represent the earth and show that a 30° central angle would intercept a very long arc on a meridian. The arc would be 1/12 the distance around the earth or about 2000 miles.

Enrichment: Discuss the definition of latitude which involves central angles. Sketch on the blackboard;



Now, have the students draw two equal circles with 50° central angles. They will agree that this time the arcs are equal (congruent).

The students should now examine their two drawings - two unequal circles and two equal circles. They should be led to realize that an arc degree is a fractional part $\left(\frac{1}{360}\right)$ of a circle. Thus, a 30° (30 angle degrees) central angle intercepts an arc which is referred to as a 30° arc (30 arc degrees). The arc degree refers to the relative size of the arc as a fraction of the circle and it does not refer to the length of arc. Thus, two arcs can have the same number of arc degrees, but not be the same length.

Also, two arcs can have the same length without having the same number of arc degrees. Therefore, two arcs may have the same measure (in degrees or in linear units, e.g. inches) and still not be congruent. However, for two equal circles, equal arcs will also be congruent arcs.

The arc degree is a new unit of measure and is used (a) in comparing the length of arc with the circumference and (b) in relating a central angle with its arc. When we wish to note that two arcs are equal in degrees, not in length, we will use the symbol $\circeq$.

The students can be led to the decision that the following should be postulated:

Postulate: In the same, or equal circles, if two central angles are equal, then their arcs are equal.

Postulate: In the same or equal circles, if two arcs are equal, then their central angles are equal.

Enrichment for those who teach similarity before circles: Use the definition of a circle and the distance formula to derive the equation of the circle.

LESSON 2

Aim: to explore another method of deducing that arcs are equal
to select the most advantageous method in a particular situation

Development:

Experiment: Divide a circle into six equal arcs. Some students will use six equal central angles because of the postulate chosen the previous day. Others will assume a new idea of equal chords and lay off the radius as a chord. They have done this in earlier grades, but were not aware that they were laying off a chord. The fact that it would take more than six radii if they were wrapped along the curve instead of laid off as chords is vital. It is necessary to draw attention to the fact that when chords are laid off, the students are assuming the proposition:

In the same or equal circles, if the chords are equal, then their arcs are equal.

The class should readily suggest that this proposition should not be postulated since it is easily deducible from a previously accepted postulate. After oral proof, the class can add, in their notes on methods of deduction, a new list:

Ways to deduce that arcs are equal:

deduce that their central angles are equal
deduce that their chords are equal

The converse of the new theorem can readily be deduced as an exercise and listed as a theorem. Ask the class, "What is the value of this new theorem in a postulational system?" They should respond that it gives them a new way to deduce that line segments are equal. Add to that list:

If they are chords in a circle, deduce that their arcs are equal.

LESSON 3

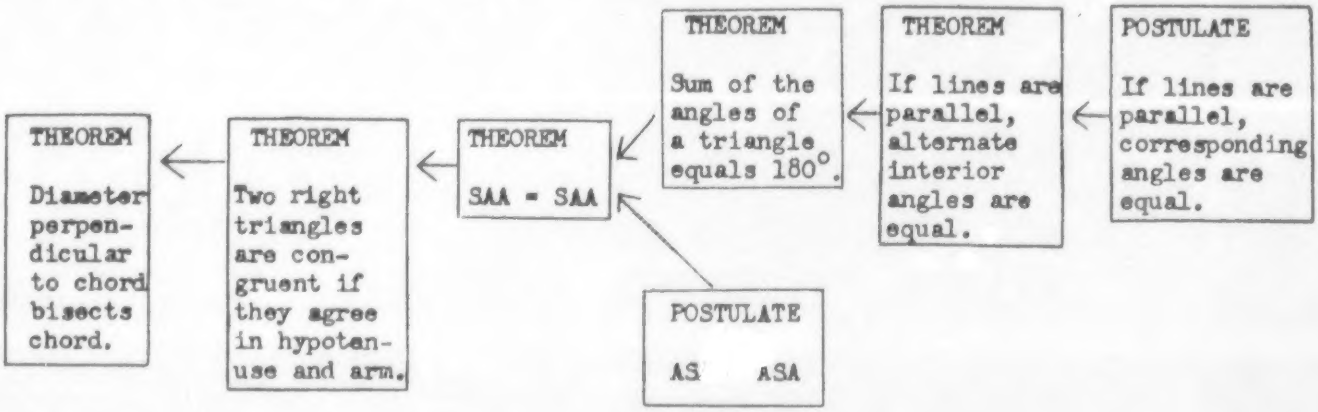
Aim: to reinforce the understanding of sequence by developing the last theorem in the first of the 4 sequences of proofs listed in the syllabus
to add a theorem very useful for further explorations and deductions

Development:

Experiment: Draw any circle and any chord CD. Drop a perpendicular from the center of the circle to CD. What do you discover? The students will quickly formulate the proposition that a line from the center, perpendicular to a chord, bisects the chord and both intercepted arcs.

The deduction should be developed orally and then written. Careful attention should be paid to selecting the best method from the list of ways to deduce that line segments are equal. Many students think that the two parts of the chords are chords themselves.

The need for the use of the hypotenuse arm method of proving congruence should lead to a reexamination of the sequence of which this theorem is the last.


THEOREM
Diameter perpendicular to chord bisects chord.
THEOREM
Two right triangles are congruent if they agree in hypotenuse and arm.
THEOREM
SAA = SAA
THEOREM
Sum of the angles of a triangle equals 180°.
THEOREM
If lines are parallel, alternate interior angles are equal.
POSTULATE
If lines are parallel, corresponding angles are equal.
POSTULATE
AS ASA

LESSON 4

Aim: to stimulate the imagination in investigating a new theorem and various rearrangements of it
to apply it to numerical exercises

Development:

Experiment: Find the center of this broken wheel.

The method selected by many students will be an introduction to the variations of the theorem explored in lesson 3. The theorem involved four elements:

passes through the center	A
perpendicular	B
bisects chord	C
bisects arcs	D

Notice that the original theorem of Lesson 3 can be written as, "If A and B, then C and D."

Students will enjoy listing all possible variations in this symbolic way, such as, "If A and C, then B and D," etc. Then have the students translate their symbolic propositions into English.

The experiment uses B and C as hypothesis and A as conclusion.

Two of the important variations are:

If a line bisects a chord and its arcs, it passes through the center of the circle and is perpendicular to the chord.

If a line through the center bisects the arcs of a chord, it is perpendicular to the chord and bisects it.

The proofs can be analyzed informally with one assigned to each row of students. They will then report to the class as a whole. It is necessary to show that to prove a line is a diameter, we must show that it divides the circle into two equal arcs.

Note that the construction for finding the missing center of a circle is an application of the locus theorem:

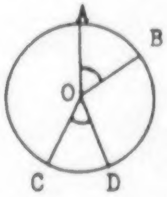
> The locus of all points equidistant from the ends of a line segment is the perpendicular bisector of that line segment.

Note: This locus theorem was introduced earlier in the year (after hypotenuse arm) and this is a good place to review the idea since students find locus ideas difficult.

TEST

1. State the theorem or postulate to justify the conclusion and label your answer as a theorem or a postulate. (Parts marked equal are given as equal.)

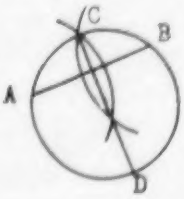
a.



Conclusion:

$\overset{\frown}{AB} = \overset{\frown}{CD}$

b.



Conclusion:

CD passes through the center of the circle.

2. Arrange in the sequence in which they were accepted or proved, and classify each as a definition, postulate, or theorem.

   a. If a diameter is perpendicular to a chord, it bisects the chord.
   b. If 2 triangles agree in ASA, they are congruent.
   c. If two right triangles agree in hypotenuse arm, they are congruent.
   d. If 2 triangles agree in SAA, they are congruent.
   e. The sum of the angles of any triangle is 180°.

3. Describe the process of finding the center of a given circle in terms of the locus theorem involved.

4. An original proof in which arcs are to be deduced equal. (See any standard text.)

5. An original proof in which chords are to be deduced equal. (See any standard text.)

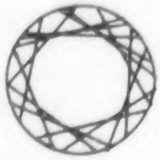
Lesson 5

Aim: To establish the relations between chords and their distances from the center.
To add a new way to deduce that line segments are equal.

Development:

Experiment: Have each member of the class draw as many chords of fixed length as he wishes in a circle.

The appearance of an inner circle will please and surprise students. (Some will enjoy doing the same with curve stitching at home.) A curve formed without drawing it by an infinite set of lines is called an envelope.



The mathematical implication that these chords are all equally distant from the center is reinforced if the teacher draws a diagram on the board with a longer chord spoiling the picture. It will be necessary to review what is meant by distance from a point to a line.

Have the students deduce informally and list as theorems:

> If chords are equal, then they are equally distant from the center.
> If chords are equally distant from the center, then they are equal.

Add to the list of ways to deduce that lines are equal:

> If they are chords of a circle, deduce that they are equally distant from the center.

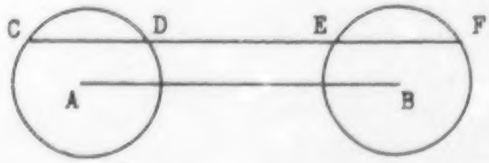
Note: If similarity is taught before circles, exercises should include many numericals in which the Pythagorean Theorem is applied to distances from the center of the circle.

LESSON 6

Aim: to apply the new ways (see lesson 5) to deduce that line segments are equal
to analyze original exercises which require the students to make a wise selection of ways to deduce line segments equal in circles

Development:

Suggested exercises:

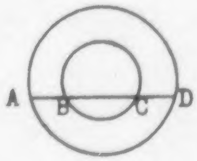


1. Given: Circle A = Circle B
   AB ∥ CF

   Deduce: CD = EF

   Plan: Show the line segments are equidistant from the centers.

Note: This exercise also serves as a good review of ways to show a quadrilateral is a parallelogram.



2. Given: Concentric circles

   Deduce: AB = CD

   Plan: Use diameter perpendicular to chord and subtract.

3. See standard textbook for other exercises.

Lesson 7

Aim: To introduce the tangent.
To develop the postulate: "If a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of contact."
To develop the theorem, "If a line is perpendicular to a radius at its outer extremity, then it is a tangent to the circle."

Development:

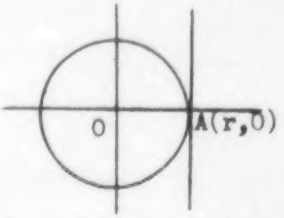
Have the students consider mud or sparks being thrown off tangent to a wheel. They should arrive at the definition:

A tangent to a circle is a line which has only one point in common with the circle.

Experiment: Have the students draw a circle and lines which they consider to be tangent to the given circle. Have them draw the radii to these lines. The students will agree that the lines appear to be perpendicular to the radius drawn to the point of contact. They may then postulate:

If a line is a tangent to a circle, then it is perpendicular to the radius drawn to the point of contact.

Experiment: Erect a perpendicular to radius OA at point A. The students will agree that this perpendicular will have only one point in common with the circle, and is, therefore, tangent to the circle.



Note: The students may want to postulate that "a line perpendicular to a radius at its extremity is tangent to the circle." However, the statement may be deduced.

OA is the shortest distance from O to the tangent because it is the perpendicular. Any other line from O to the tangent, being longer than OA, must meet the tangent outside the circle.

If similarity is taught before circles, the following proof by coordinate geometry could be used instead.

Given: Circle O whose equation is $x^2 + y^2 = r^2$.

A line perpendicular to radius OA where A (r,o).

Since the line is perpendicular to the X-Axis, its equation is $x = r$.

Solving the equations of line and circle simultaneously:

$$x^2 + y^2 = r^2 \qquad r^2 + y^2 = r^2$$

$$x = r \qquad y^2 = 0$$

$$y = 0$$

The only point of intersection is (r,0). Therefore, the line is tangent.

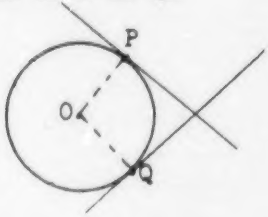
LESSON 8

Aim: to develop several theorems concerning tangents
to apply these to exercises

Development:

Experiment: Construct a tangent to circle O at P and also at Q.

What do you discover?



The discussion concerning the resulting kite figure will bring out a number of easily deduced ideas:

The tangents are equal. (Define "length of tangent" as distance from outside point to point of contact.)
The opposite angles are supplementary.
The line to the center bisects the angle between the tangents.
The line to the center is the perpendicular bisector of the chord.
(Review again the idea of locus.)

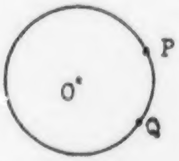
These are to be listed as theorems after informal proof.

What happens as we move the point of intersection of the tangents closer to the circle? Farther away?

If similarity precedes circles, have students now do exercises in which the Pythagorean Theorem and equal tangents are involved.

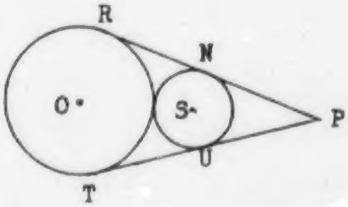
TEST

1. a. Select P and Q so $\angle POQ$ will equal 60°.
   b. Construct tangents to the circle at P and Q, meeting at M.
   c. Find $\angle PMQ$.
   d. Deduce that OM is the perpendicular bisector of PQ.



2. Given: RP and TP tangent to circles O and S

   a. Deduce: RN = TU



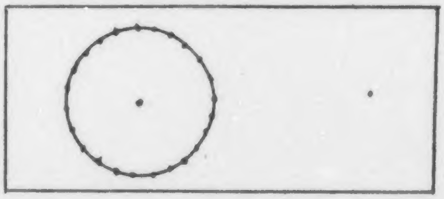
3. An exercise in proving chords equal. See any standard text.

LESSONS 9 and 10

Aim: to introduce angle measurement in circles
to examine the relationships between angles in a circle and their arcs

Development:

Note: It is advantageous, when possible, to present such a topic experimentally. The students should discover certain propositions, note their significance and application, and try to arrange the propositions in a deductive sequence.



Have available a wooden model of a circle with nails projecting from it, perpendicular to the plane of the circle, and equally spaced as shown in the diagram at the right. Nails should be placed every 15° so that the students can read off the number of degrees in an arc.

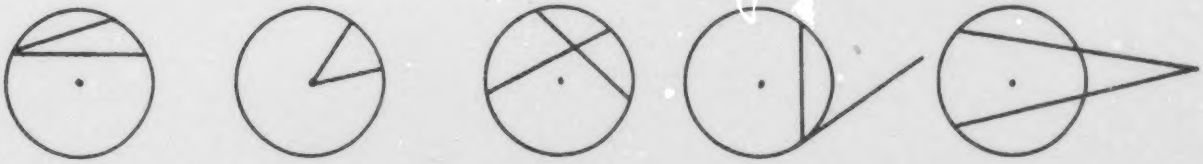
It is also possible (but less effective) to trace a blackboard protractor on the board.

If your school has a polar coordinate stencil graph chart, it has 360 dots in the outer circle making it very easy to read off degrees in any arc.

Have the students recall that various arcs of 30° (30 arc degrees) may be unequal in length. (See lesson 1 of this chapter.) The symbol $\overset{a}{=}$ indicates arcs are equal in arc degrees, but not necessarily in length. Compare this to "James $\overset{h}{=}$ John" meaning "James and John are the same height."

Have the students recall that a 30° central angle (30 angle degrees) intercepts an arc which is $\frac{1}{12}$ of the circumference or is referred to as a 30° arc (30 arc degrees). Is there always a relationship between angles formed by lines in a circle and their arcs? Let us investigate various possibilities.

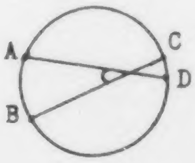
Class will suggest various possibilities. Show these possibilities with rubber bands stretched between nails on the wooden model.

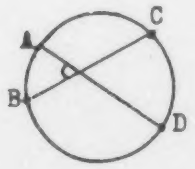


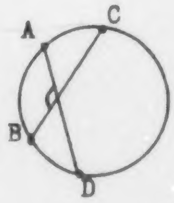
In each case, have the students come to the model, measure the angle with a protractor, and read off the arcs by means of the nails. List the relationships as they are discovered (in any order). Class should suggest trying at least 3 different positions of the lines in each case. Mention the need for a fair sampling.

The angle formed by two chords intersecting within the circle presents an interesting challenge. The results might appear as:

$\angle$	$\widehat{AB}$	$\widehat{CD}$
30°	45°	15°
60°	45°	75°
128°	75°	180°







The results should suggest averaging the arcs. The idea of an arithmetic mean has already been studied in the coordinate geometry unit on the midpoint of a line segment.

By the end of lesson 10, the class should have compiled a list of the relationships which they have discovered:

> An angle formed by two chords intersecting inside the circle is equal in degrees to the average of its arcs.
>
> An angle formed by two secants (tangent secant, two tangents) is equal in degrees to half the difference of its arcs (intercepted).
>
> An angle formed by a tangent and a chord is equal in degrees to half its arc.
>
> An inscribed angle is equal in degrees to half its arc.
>
> A central angle is equal in degrees to its arc.

LESSON 11

Aim: to apply the angle and arc relationships discovered in lessons 9 and 10.

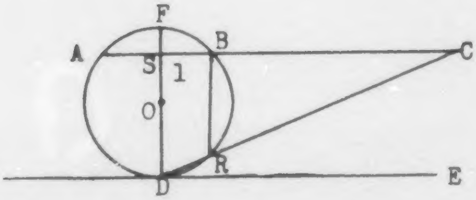
Development:

Note: Exercises, including many which involve some use of algebra, may be found in any standard text and in previous New York State Regents examinations. These exercises also will provide a review of many principles introduced earlier. Exercises involving Pythagorean Theorem and trigonometry should also be provided, if similarity precedes circles.

Suggested exercise:

DF is diameter of circle O
DE is a tangent, $DE \parallel AC$
$\widehat{BF} = 40^\circ$, $\angle C = 20^\circ$

Find: $\angle 1$, $\widehat{BR}$, $\angle SDC$, $\angle CDE$, $\angle SBR$



LESSONS 12 and 13

Aim: to rearrange the angle and arc relationships discovered in lessons 9 and 10 into a postulational sequence

Development:

Have the students consider which of the five relationships would be best chosen as a postulate. The students can be guided to see that - "A central angle is equal in degrees to its arc" - is the relationship which should be selected as the postulate since it seems to contain the minimum assumption. Whatever facts are given, the center will be one of them.

An examination of the diagrams involving angles and circles will reveal that the relationships seem to be easily associated with the inscribed angle. Therefore, this theorem should be deduced next after the postulate. It will probably be necessary to suggest the separation of the inscribed angle theorem into three cases, since this is the first time students have met this device. Students may argue that this is reasoning from a special case. The teacher should point out that this is acceptable since every possible case is discussed. The usual error is precisely an error in not considering every possible case.

Note: Students may suggest other orders and, for enrichment, they may try them at home.

The deductions in this sequence illustrate a simple postulational system. They may be called for on the Regents examination. The four required deductions should be written by the students at home using letters other than those used in class. These deductions should be carefully checked.

Note: The tangent chord theorem is a side branch of the sequence and its deduction is not listed as 'required', but it should be done informally.

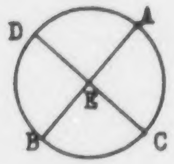
LESSON 14

Aim: to show that these angle and arc relationships are in reality all special cases of one general formula:

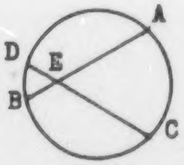
An angle formed by two lines in a circle is equal in degrees to the average of its arcs.

Development:

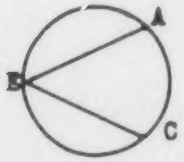
Use the wooden model or diagrams. Have the students realize that the central angle is half the sum of two equal arcs (average of the arcs).



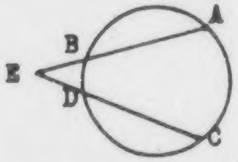
As E (point of intersection) moves away from the center, one of the arcs (BD) decreases.



The arc (BD) becomes zero when E is on the circle.



The arc (BD) reappears as E moves outside of the circle. Students will often suggest that the arc BD which decreased to zero and now reappeared should be considered negative. Thus, the one formula will apply to all cases. (The tangent chord and secant tangent should also be included.)

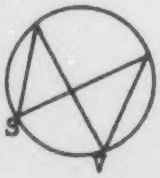


LESSON 15

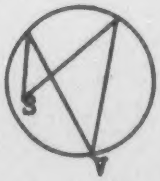
Aim: to add to the lists of "Ways to deduce that arcs are equal" and "Ways to deduce that angles are equal," the new ways arising from the angle measurement sequence
to deduce that an angle inscribed in a semicircle is a right angle
to apply these to exercises

Development:

Have the students consider $\angle S$ and $\angle V$ in the diagram at the right. They will be able to deduce that the angles are equal.



Have them next consider $\angle S$ and $\angle V$ in the diagram at the right. Is $\angle S = \angle V$?



The following theorem should be listed:

Inscribed angles measured by the same or equal arcs are equal.

Many cases can be handled algebraically. The angles will be equal when shown to be equal to the same expression.

These theorems should be added to "How to deduce..." lists.

Have them deduce that an angle inscribed in a semicircle is a right angle.

LESSON 16

Aim: To develop two ways of constructing a tangent to a circle.

Development:

Experiment: Point P is on circle O. Construct a tangent to O at P. (This is a direct application of the theorem that a line perpendicular to a radius at its outer extremity is tangent to the circle.)

Experiment: Point P is outside circle O. Construct a tangent to O from P. This is a real challenge and a good place to teach the method of analyzing a construction by sketching the finished product. Students will suggest that to get the necessary right angle at the point of contact, they need a semicircle with OP as diameter.

UNIT TEST

CIRCLES

1. Arrange these theorems, definitions, and postulates in the sequence in which we accepted and deduced them:

   a. The sum of the angles of a triangle is 180°.
   b. The angle between two chords is equal in degrees to half the sum of its arcs.
   c. A central angle is equal in degrees to its arc.
   d. An exterior angle of a triangle is the sum of the remote interior angles.
   e. An inscribed angle is equal in degrees to half its arc.

2. Deduce: An angle between a tangent and a secant is equal in degrees to half the difference of its arcs.

3. Discuss the method we used to deduce the "Inscribed angle" theorem, and explain why the method used is not an error in reasoning (reasoning from a special case)

VIII.

SIMILARITY

LESSON 1

Aim: to define similar polygons
to define and use ratio and proportion
to develop that in a proportion the product of the extremes equals the product of the means

Development:

This lesson may be introduced with a reference to the fact that the class up to now has given considerable attention to the study of congruent polygons - polygons which have the same size and the same shape. Have the students note that the symbol $\cong$ consists of two parts: $=$ representing the same size, and $\sim$ representing the same shape. It is now proposed to consider each of the two facets of congruence separately. In this unit we consider figures which have only the same shape (called similar figures) and in a later unit we will study figures which have only the same size (that is, are equal in area).

Have the students suggest situations in which they have encountered similar figures (scale drawings, photo enlargements, blueprints, maps, scale models, and so on).

What mathematical relationships appear to exist between the corresponding parts of similar figures? In order to describe the relationship, the word ratio will be introduced. Its definition as the quotient representing the relative size of two quantities should be introduced at this point. To develop the meaning of ratio, the students should be asked to find the ratio in such exercises as the following:

number of boys to the number of girls in the class
number of girls to the number of boys in the class
number of boys to the total number of students in the class
2" to 1'

To have students arrive at a precise definition of similar polygons, the following experiment should be performed:

> Draw a rectangle with length 2" and width 1"
> Construct a figure of the same shape but having its length 3".

What two relationships between parts was it necessary to arrange in order to secure the same shape? Would every rectangle be similar to every other rectangle since their angles are all equal? Would a rectangle with 1" and 2" sides be similar to a parallelogram with $1\frac{1}{2}$" and 3" sides?

The definition of similar polygons as polygons in which (1) corresponding angles are equal and (2) the ratios of corresponding sides are equal, should then follow.

Note: In arriving at the definition of similar polygons, the teacher will find a need for a much more careful development of the notion of corresponding angles and corresponding sides than was the case with congruent triangles. If it is possible to name two polygons ABCDE... and A'B'C'D'E'... in such a way that $\angle A = \angle A'$, $\angle B = \angle B'$,..., then $\angle A$ and $\angle A'$, $\angle B$ and $\angle B'$,... are what we mean by corresponding angles, and AB and A'B', BC and B'C',... are what we mean by corresponding sides.

The definition of similar polygons given here contains more information than is necessary. Occasionally, a bright student may point out that he needed only to construct two of the angles and three of the sides, or two of the sides and three of the angles in his new rectangle to insure similarity. He should be told that convention has dictated the adoption of this definition to effect an economy of language description rather than an economy of mathematical requirements. From the conventional definition it will be deduced later that fewer properties are needed to show polygons similar. If this issue does not arise spontaneously at this time, it would be wise for the teacher not to inject it. Later studies (the constancy of the angle sum for a polygon of a fixed number of sides) will provide opportunities for appropriate questions on how their implications will affect the definition of similar polygons.

The equal ratios of the sides of similar polygons leads naturally to the definition of a proportion as the statement that two ratios are equal. Students frequently confuse the terms ratio and proportion in using them. It should be emphasized that a ratio compares two terms while four terms are involved in a proportion. The number of boys to girls in our class is a certain ratio. If the ratio of the number of boys to girls in the class next door is the same as in ours, then boys and girls of both classes are in proportion.

The terms means, extremes, and fourth proportional should be introduced at this point. They are best explained from a proportion written in the form $a:b = c:d$, but students should be told that $\frac{a}{b} = \frac{c}{d}$ is usually a better form with which to work.

Have the students examine numerical proportions ($2/4 = 6/12$) to discover the relation of means to extremes. This should lead to the generalization that in a proportion, the product of the means equals the product of the extremes. This law should be deduced by getting the class to suggest multiplying both members of the generalized proportion $\frac{a}{b} = \frac{c}{d}$ by bd.

The law should then be applied in numerical and algebraic situations. Suggested exercises:

1. Is $\frac{3}{7} = \frac{4}{9\ 1/3}$ a true proportion?

2. Find the fourth proportional to 3, 5, and 12.

3. Find x: $\frac{x - 3}{10} = \frac{4}{x}$.

LESSON 2

Aim: to postulate the proportional division of two sides of a triangle by a line parallel to the third side
to use the postulate in developing the transformations of proportions

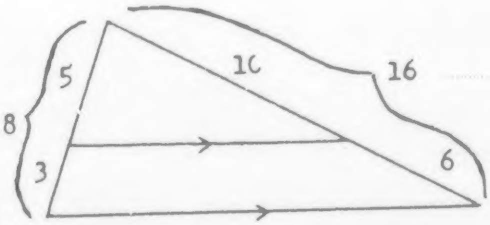
Development:

Experiment: Have the class draw a triangle (not isosceles) with a line parallel to one side of the triangle intersecting the other two sides. The four segments into which the intersected sides are cut should then be measured (use a millimeter scale to do this to avoid the difficulties that result from the fractional parts when the inch scale is used). Is there any apparent relation between the four segments?
Note: The experiment is easily done on the overhead projector with the teacher indicating the readings on a transparent millimeter ruler or a homemade acetate marked straight edge.

From the experiment, students should arrive at the generalization that "If a line is parallel to one side of a triangle and intersects the other two sides, it divides them proportionally." We have no previously established method for deriving a proportion, other than from the sides of similar polygons (where the proportion is needed to establish the similarity). Therefore, we cannot deduce this generalization from anything previously established and must list it as a postulate.

Note to teacher: In fact, this proposition is even more of an assumption than we have brought out here. The whole idea that any two lines could be measured, using the same unit, if only the unit chosen were small enough, was at first assumed by the Greeks, and the fact that this cannot be done when the ratio of the lengths is irrational (incommensurable) came as a shock to the Pythagorean School.

A new list, "Ways to deduce lines proportional," should now be started. "Showing that they are corresponding sides of similar polygons" and "showing that a line cutting two sides of a triangle is parallel to the third side" are entered in this list.



Have students consider a numerical application of the above (see diagram). Clearly, $5/3 = 10/6$. Are there any other true proportions connecting the segments? By experimentation with these numbers students will find that $5/10 = 3/6$, $5/8 = 10/16$, and so on. It is also instructive to have pupils consider cases that "do not work" ($5/10 \neq 6/3$ and $5/6 \neq 10/3$). This will lead them to formulate the need for a correspondence in location of those segments whose sizes are shown in corresponding positions in the equal ratios. By labeling the segments a, b, c, and d, the numerical examples above should be generalized as:

(1) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{b}{a} = \frac{d}{c}$.

(2) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{c} = \frac{b}{d}$.

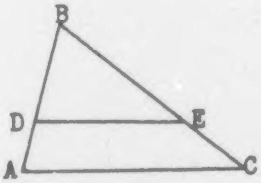
(3) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a + b}{b} = \frac{c + d}{d}$.

If time permits, brief informal proofs can be given for the above. In (1) multiply both sides by bd and then divide both sides by ac. In (2) take product of means and extremes and divide by cd. In (3) add 1 to each side of the original.

Note: In view of the above transformations, it is not necessary to distinguish between the sides of a triangle being divided "in proportion" and "proportionally."

Have the students do numerical exercises in which the "line parallel to a side of a triangle" postulate is involved. Exercises such as the following may be found in any textbook.

If BD = 6, DA = 3, BC = 12 and DE is parallel to AC, find BE.

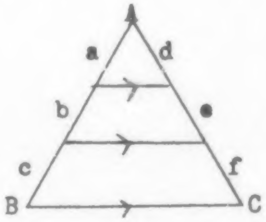


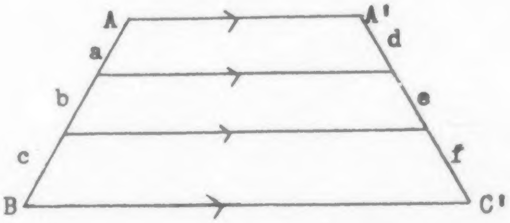
LESSON 3

Aim: to develop that three or more parallel lines divide transversals proportionally
to develop that three or more parallel lines cutting off equal segments on one transversal cut off equal segments on every transversal
to divide a line into any number of equal parts

Development:

Have the students draw triangle ABC with two lines parallel to BC instead of one as in lesson 2. What relationship between a, b, c, d, e, and f would you expect? Verify by measuring.





Have the students imagine triangle ABC is "stretched" by moving AC parallel to itself until it assumes the position A'C'. a, b, c, d, e, and f are not altered in size. Class discussion will lead to the proposition "three or more parallel lines intercept proportional segments on two transversals." This postulate should now be added to the list of "Ways to deduce a proportion." If you prove the proposition, list it as a theorem.

Now have the students suppose that a, b, and c of the figure are all equal. What must follow about the sizes of d, e, and f? The corollary that "If three or more parallel lines cut off equal segments on one transversal, they cut off equal segments on every transversal" should be elicited. This should now be added to the list of "Ways to deduce line segments equal."

Experiment: Have the students take a piece of paper with an edge about 4" long, and divide this edge into 5 equal parts without using ruler or compasses. Have them measure the third angles and the sides, and note that the corresponding angles have been forced to be equal, and the corresponding sides to be proportional by merely copying two angles. The generalization that, "It seems as if two triangles are similar if they agree in two angles," can now be made. (Hint: You may use any lined paper you have in your notebook to help you.) When this is solved, the teacher should elicit the two properties possessed by the lined paper that made it possible to do this (the lines are parallel and they are equally spaced from one another).

The results from the above experiment should now be applied to dividing a line segment into any number of equal parts using straightedge and compasses. The students should be capable of developing this construction themselves if the teacher's questioning leads them to appreciate the need for the equally spaced parallel lines and that the endpoints of the segment to be divided must lie on the first and last of these lines.

LESSONS 4, 5 and 6

Aim: to discover methods for making triangles similar
to deduce the simplest method (angles equal)
to apply this theorem in deducing triangles are similar
to apply this theorem in deducing proportions
to apply this theorem in deducing the equality of products
to deduce certain corollaries

Development:

Have the students review the definition of similar polygons. They should now be asked to draw a triangle and to consider the problem of making a triangle similar to it. Since the definition involves equal angles and proportional sides, a third of the class can be assigned to begin their attack on the problem by making angles equal (with protractors or compasses); a second group should begin to work by doubling corresponding sides; and a third group should try some combination of equal angles and proportional sides.

It will become apparent to the first group that the required triangle is fully determined as soon as two angles are made equal to those in the original triangle and that there is no need to construct the third angle or to take any steps to insure that the sides are in the same ratio. The generalization that "Two triangles are similar if they agree in two angles" can now be made.

With the help of some questions by the teacher, the second group can also be led to see that their triangle is fully determined when they have its sides, and therefore there is no need to attack the problem of making the angles equal. What generalization results? Finally, the last group can see that the construction of one angle and its including sides will also determine a triangle that meets the requirements.

Students should be asked to compare this situation involving similar triangles with that of congruent triangles. The definition of congruent triangles requires the equality of six pairs of parts, but it was deduced that the equality of three pairs of certain parts would suffice to establish congruence.

Which of the three new methods for constructing similar triangles appears to be simplest? Since the first method involves the fewest elements, the question can be raised as to whether all the conditions in the complete definition of similar triangles can be deduced from the equality of just two pairs of angles. It should be emphasized to students that the equality of angles alone will not guarantee similarity of polygons other than triangles (compare a square and a non-square rectangle, for example).

In attempting to deduce the AA = AA proposition, students will readily see why the third pair of angles are equal. They should now note that we have only one previously established way to deduce the proportionality of sides. Manipulation of a pair of similar triangles cut out of cardboard will suggest the placement of triangles so that a line is parallel to one side of a triangle. The deductive proof then follows.

Note: The proof required in the New York State syllabus as well as that given in many texts is the one for AAA = AAA, rather than for AA = AA. Students should be asked how they would modify their proof if they were asked to prove the former proposition.

Two pairs of angles equal can now be placed in the list, "Ways to deduce triangles are similar."

Have the students do simple deductive exercises proving triangles similar using AA = AA. One of these exercises should be the deducing of the corollary that "A line parallel to one side of a triangle cuts off a triangle similar to the original triangle." This proposition should be added to the list of "Ways to deduce that triangles are similar."

Enrichment: The corollary can also be used to explain the principle which is used in the pantograph to construct similar figures.

Have the students now consider exercises in which it is required to deduce line segments proportional. In order to get a correct proportion, corresponding sides of similar triangles must be selected. It should be stressed that corresponding sides are opposite equal angles. Marking the equal angles on the figure as they are proved equal is therefore a "must." Since two pairs of equal angles will be marked in using AA = AA, the remaining unmarked pair also represent a set of corresponding angles.

Students may also have a problem in determining which triangles to select to prove similar in order to deduce a particular proportion. Since two sides of one of the triangles appear in the numerators, and two sides of the other appear in the denominators of the proportion, the vertices of the required triangles are determined by the letters in the numerators and denominators respectively.

Note: In the case of a "transformed" proportion, two sides of the same triangle may appear in one ratio, and the corresponding sides of the other triangle in the other ratio. In this case, the vertices in each ratio will determine the names of the triangles to be deduced similar.

The students should be led to understand the value of analyzing the proportion to be deduced. They should write a plan indicating which triangles have been selected to be proved similar. It is also good practice to have the students draw their diagrams in pencil and outline the selected triangles either with two different colored pencils or in ink. They might use a solid ink line for one triangle, and a dotted ink line for the other.

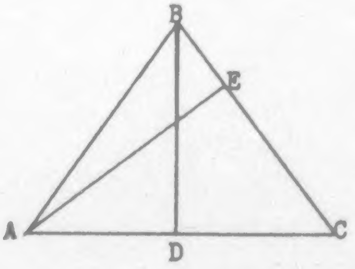
Suggested exercise:

Given: △ ABC with altitudes BD and AE

Deduce: $\frac{AC}{BC} = \frac{AE}{BD}$

Plan: (1) △ ACE ~ △ BCD to be deduced by (Student then outlines these △ s)

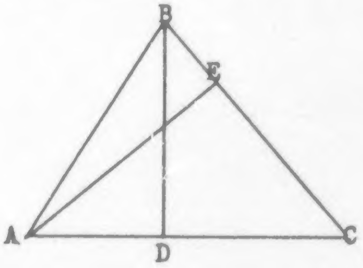
(2) AA = AA



As another exercise, the students should be asked to consider the relationship of corresponding altitudes in similar triangles. It should be deduced that they are in the same ratio as any two corresponding sides, and this fact added to "Ways to deduce that lines are proportional."

The exercises in which line segments were proportional should next be extended to involve the deduction that two products are equal. Presentation of such exercises without any special preparation of the class will usually elicit the student suggestion that such an equality of products will result if the product of the means and product of the extremes are taken in a proportion. The proposition that "If the product of two numbers is equal to the product of another two numbers, then either pair can be made the means and the other pair the extremes of a proportion" can be shown to hold by dividing both sides of $ab = cd$ by bc, and so on.

Students should apply this proposition as part of the analysis of exercises involving equal products. The analysis results in a plan to deduce a certain proportion, a certain pair of triangles are similar, and usually that the triangles are similar because of AA = AA.



Suggested exercise:

Given: $\triangle ABC$ with altitudes BD and AE

Deduce: AC x BD = AE x BC

Plan: (1) $\frac{AC}{BC} = \frac{AE}{BD}$ to be deduced by

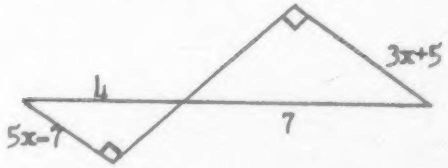
(2) getting $\triangle ACE \sim \triangle BCD$ to be deduced by (Student then outlines these $\triangle$ s)

(3) AA = AA

Have the students consider numerical and algebraic exercises involving computation to find the length of the sides of similar triangles. These exercises should include some involving indirect measurement by similar figures and some applications to scale drawings.

Suggested exercises:

1. Find the height of a tree if it casts a shadow 70' long at the same time that a 5' pole casts a 7' shadow.
2. Find the value of x in the diagram to the right.



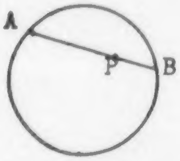
3. In a scale drawing, a 3" line is used to represent a side of a triangle that is actually 20' long. What length should be used in the drawing to show a second side of the triangle which is 35' long?

LESSONS 7 and 8

Aim: to study relationships between segments of chords; secants, and tangents

Development:

Experiment: P is any point on AB.
As AB rotates about P, what
happens to its segments?



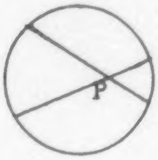
Students will readily note that as one segment of the chord grows, the other shrinks. Measure two cases and note that the product is constant.

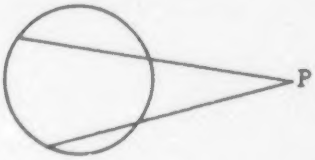
> Teacher: Is it necessary for the product to be constant just because one grows when the other shrinks?
> Student: Not unless they are in proportion.

This suggests similar figures, and the informal proof is readily obtained. List theorem:

> If two chords intersect within a circle, the product of the segments of one equals the product of the segments of the other.

What would be the effect of pulling
P outside the circle?





Students may prefer deduction to experiment this time. Prove informally and list theorem:

> If two secants meet outside a circle, the product of one secant and its external segment is equal to the product of the other secant and its external segment.

Suggested exercises:

Apply both theorems to algebraic exercises.

1. A secant is drawn to a circle from an outside point and has an internal segment of 5 inches. Another secant from the same point has an internal segment of 9 inches and an external segment of 3 inches. Find the external segment of the first secant.
2. In a circle, chords AB and CD intersect at E. If AE = 10, EB = 4, and DE exceeds EC by 3, find EC.

Move P so that one of the secants becomes a tangent. What happens to the segments? Theorem should therefore become:

"If a tangent and a secant are drawn to a circle from the same external point, the square of the tangent is the product of the secant and its external segment."

Deduction is assigned as an original exercise:
Suggested exercise: A tangent and a secant are drawn to a circle from an external point. If the tangent is 9, and the internal segment of the secant is 24, find the external segment of the secant.

LESSONS 9 and 10

Aim: to discover the converse of the proposition concerning a line parallel to one side of a triangle
to use the converse to deduce that "two triangles are similar if an angle of one triangle is equal to an angle of the other triangle and the sides including these angles are in proportion"
to deduce "a line joining midpoints of two sides of a triangle is parallel to the third side and equal to one half of it"
to apply the theorems to exercises.

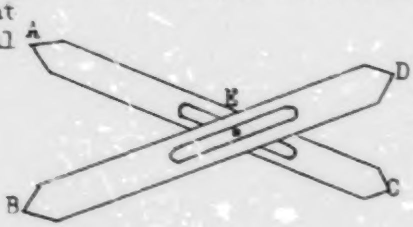
Development:

Experiment: Take a triangle (preferably non-isosceles) and divide two of the sides (by inspection) into 3 or 4 equal parts. Connect a pair of corresponding points of division so that both sides are cut in the same ratio (such as 1:2 or 1:3), and note the relationship of the joining line to the third side of the triangle. The generalization that a line dividing two sides of a triangle proportionally is parallel to the third side should be postulated. It can be added to the list "Ways to deduce that lines are parallel."

Have the students realize that line segments in proportion were used to suggest the above postulate. Raise the question of whether line segments in proportion can also be used to deduce triangles similar. Students should recall or repeat the experiment of the first lesson dealing with similar triangles. Each student drew a triangle and constructed a triangle similar to it based upon the definition of similar polygons. A group of students copied one angle and made the sides forming the angle into equal ratios (by doubling, halving, or tripling each). It became apparent that the similar triangle was determined by constructing the three parts, and thus making it unnecessary to construct any of the other relationships required in the definition.

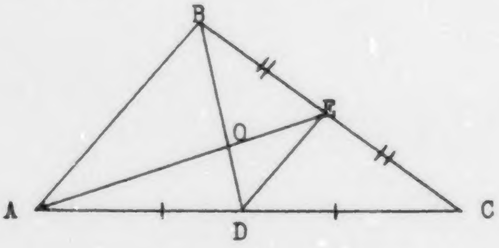
Note: The deduction of the theorem, "Two triangles are similar if an angle of one is equal to an angle of the other and the including sides are proportional," is not required in the State syllabus. However, it is advisable to do it in class since it begins in the same manner as the AA = AA deduction and will help to emphasize certain aspects of the former. (Why cannot any angle be superimposed on the corresponding one in the new proposition?)

This similarity theorem may be used to explain the principle of proportional dividers in which pin E is adjusted so that AE:EC = BE:ED, and in which a pair of equal vertical angles always occurs.



Students should now be asked to consider what facts appear to be true when a line is drawn joining the midpoints of two sides of a triangle. The proposition that "A line joining the midpoints of two sides of a triangle is parallel to the third side and equal to one half of it" can be deduced very readily from the second similarity theorem. It should be added to "Ways to prove lines parallel" and applied to both deduction exercises and to numerical exercises.

The theorem "A line joining the midpoints of two sides of a triangle..." may be used in connection with the deduction of "two medians of a triangle trisect each other." (Connect points D and E, $DE \parallel BA$ and $DE = \frac{1}{2}BA$. $\triangle DOE \sim \triangle OBA$. $OE = \frac{1}{2}OA$, and $OD = \frac{1}{2}OB$. Therefore, $OE = 1/3AE$ and $OD = 1/3\ BD$.



Suggested exercises: (See any good textbook)

1. Two isosceles triangles are similar if their vertex angles are equal.
2. If the line joining the midpoints of two adjacent sides of a rectangle is 8, how long is a diagonal of the rectangle?

LESSON 11

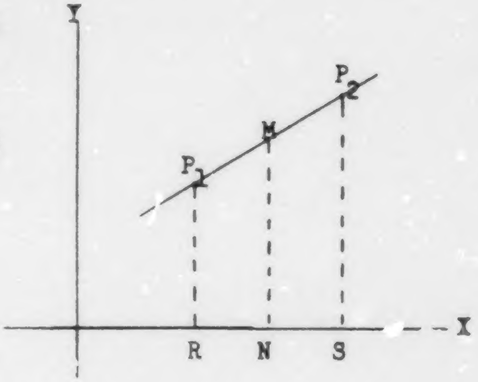
Aim: to develop "The coordinates of the midpoint of any line segment from $P_1\ (x_1, y_1)$ to $P_2\ (x_2, y_2)$ are the averages of the coordinates of the endpoints: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$"

to apply the theorem

Development:

The term "arithmetic mean" could well be used instead of "average." Modern civilization calls for an increased understanding of measures of central tendency. This topic gives students an increased facility in finding what number lies halfway between two others, especially when negative numbers are involved. Frequent oral response will discourage the tendency of some students to memorize the formula to the extent that meaning is lost. Some students will still be vague about the distinction between addition and subtraction of signed numbers. The distinction deserves frequent reexamination.

The proof of the theorem follows readily as illustrated:



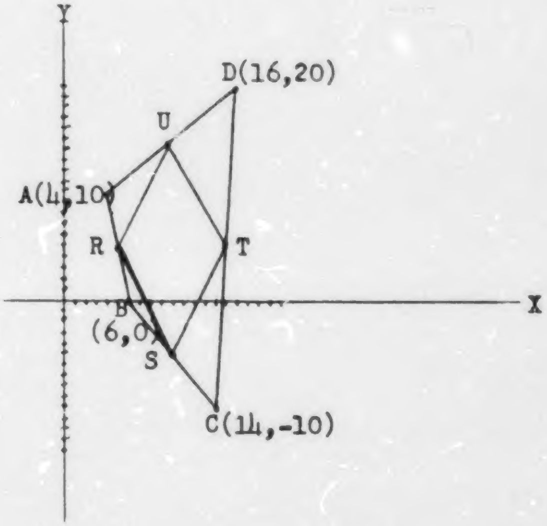
Given that M is the midpoint of P_1P_2. Drop P_1R, P_2S and MN ⊥ to the X-Axis. Therefore, N is a midpoint of RS since "If 3 or more parallel lines cut off equal segments on one transversal, they cut off equal segments on any transversal." R is $(x_1, 0)$ and S is $(x_2, 0)$.

Therefore, N is $\left(\frac{x_1 + x_2}{2}, 0\right)$. Since MN is parallel to the Y-Axis, the abscissa of M is also $\frac{x_1 + x_2}{2}$.

Repeat with perpendiculars to the Y-Axis for the ordinate $\frac{y_1 + y_2}{2}$.

Suggested exercises:

1. Plot A(4,10), B(6,0), C(14,-10), D(16,20). Find the coordinates of the midpoints of each side of ABCD. What kind of a quadrilateral is formed by joining the midpoints in succession? Why?

   Acceptable student answer:

   R(5,5), S(10,-5), T(15,5), U(10,15)

   RT and US are perpendicular and they have a common midpoint (10,5)

   Therefore, RSTU is a rhombus because "If the diagonals of a quadrilateral bisect each other, it is a parallelogram" and "If the diagonals of a parallelogram are perpendicular to each other, it is a rhombus."

   Note: Students lose sight of the significance of these exercises, whether demonstrated as above, or proved deductively (informally).

   They should be asked whether this would be true of all quadrilaterals, they should be reminded by wooden models of the flexibility of the shape of a given quadrilateral, and they should be encouraged to make a dynamic model of wood or cardboard and rubber cord. Some students get a much more lasting feeling for these ideas when they see them in three different settings - demonstrative proof, coordinate geometry exercises, and wooden models.

2. Plot A(0,0), B(a,0), C(b,c). Find the coordinates of the midpoints R and S of AC and BC respectively. How long is RS? Make two observations about RS. State these two observations as a theorem.

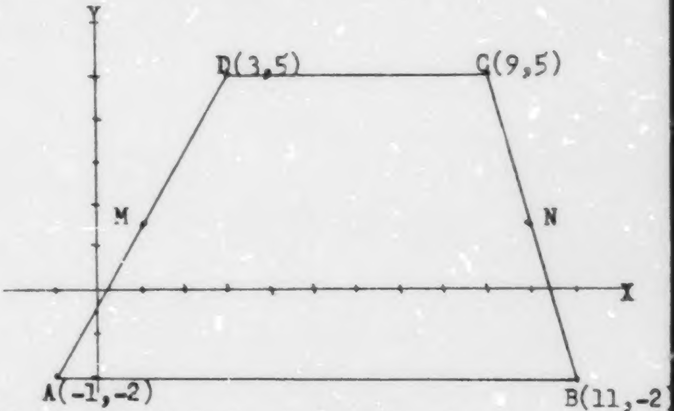
   It will be necessary to discuss the placing of the axes in the position where the origin is on one vertex, and the X-axis falls along one side. The generality of this method should be made clear. The technique is an important one for mathematics.

   Note: Earlier, students have deduced synthetically that the line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half of it. This fact is deduced here by an analytic procedure.

   The theorem is listed in the syllabus and can be given this treatment instead of the synthetic one and then listed in the students' notebooks.

   The inclusion of coordinate geometry, which some have felt may overcrowd the syllabus, does in fact, therefore, help to treat many ideas in the informal deductive manner.

3. Plot A(-1,-2), B(11,-2), C(9,5), D(3,5). Find the coordinates M and N of the midpoints of AD and BC. How long is AB, DC, MN? Discover a relationship among the lengths of AB, DC and MN.



   Acceptable student answer:

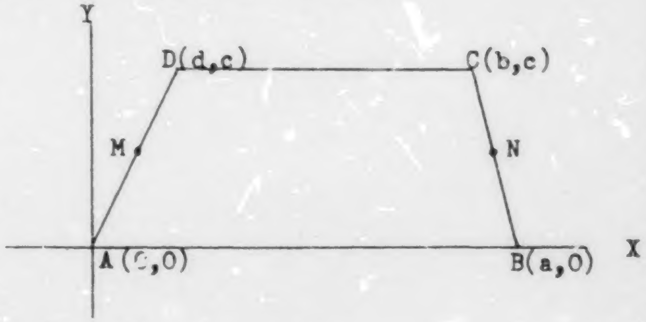
   a. M $(1, \frac{3}{2})$; N $(10, \frac{3}{2})$ AB = 12, DC = 6, MN = 9

      It seems that MN is the average of AB and DC.

      Teacher: What kind of reasoning are you doing now?

      Student: Experimental (or inductive).

   b. For some students assign coordinates A(0,0), B(a,0), C(b,c) and D(d,c) and discuss the generality and deductive nature of this method. Conclude that the median of a trapezoid is the average of the bases, and list it.



      Acceptable student response:

      M $(\frac{d}{2}, \frac{c}{2})$; N $(\frac{a+b}{2}, \frac{c}{2})$

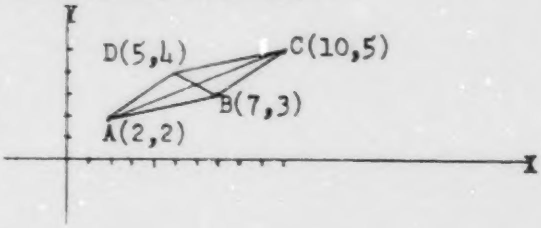
      Therefore, MN = $\frac{a+b-d}{2}$

      AB = a
      DC = b - d

      MN is equal to the average of AB and DC.
      MN is also parallel to AB and DC because all three are parallel to the X-Axis.

4. Plot A(2,2), B(7,3), C(10,5), D(5,4). Why is ABCD a parallelogram?

Students can now use the fact that the diagonals have the same midpoint even when the diagonals are not conveniently located on the coordinate grid lines.



LESSON 12

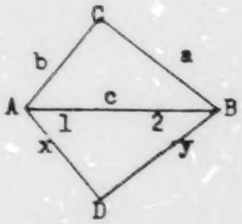
Aim: to present and deduce informally: "Two triangles are similar if their corresponding sides are in proportion "

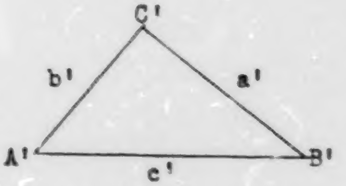
Note: This proposition is not listed in the State syllabus. Its inclusion is recommended to "round out" the discussion of similar triangles and to make the conclusion available for certain applications.

Development:

Have the students consider two triangles ABC and A'B'C' (a photograph and its enlargement, or an actual triangle and a photograph of it which is an "enlargement" or a "reduction" of it) which have $a/a' = b/b' = c/c' = k$ (k is the proportionality constant). What else is constant? Elicit from the students that the size of the angles seems to have been preserved and thus the triangles are apparently similar.

An informal demonstration of the similarity may be done as follows: Measure $\angle A'$ and $\angle B'$ and construct $\triangle ABD$ with $\angle 1$ and $\angle 2$ respectively equal to $\angle A'$ and $\angle B'$. Thus, $\triangle ABD \sim \triangle A'B'C'$ (AA = AA) so that $x/b' = c/c'$ or $x = kb'$. Thus, $x = b$. In the same way $y = a$. Therefore, $\triangle ABD \cong \triangle ABC$ (SSS = SSS) and hence $\triangle ABD \sim \triangle ABC$ (congruent triangles are similar). Then $\triangle ABC \sim \triangle A'B'C'$ since they are both similar to the same triangle





Given: $\triangle ABC$ and $\triangle A'B'C'$
$a/a' = b/b' = c/c' = k$
$(a = ka', b = kb', c = kc')$

Deduce: $\triangle ABC \sim \triangle A'B'C'$

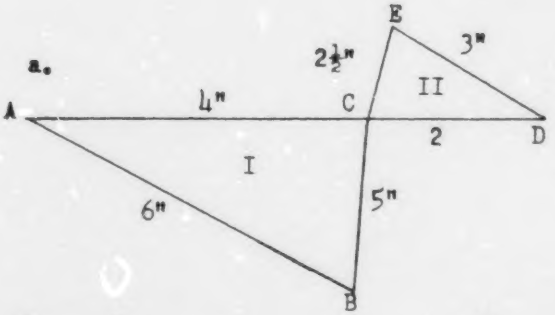
The theorem may be restated, "Two triangles are similar if the sides of one are equimultiples of the sides of the other." It should be included in the list of methods to deduce triangles similar.

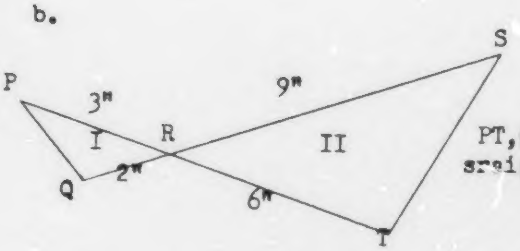
Suggested exercises:

1. How are two triangles related if the sides of one are equimultiples of the sides of the other (a) when the proportionality constant $k < 1$? (b) when $k > 1$? (c) when $k = 1$?

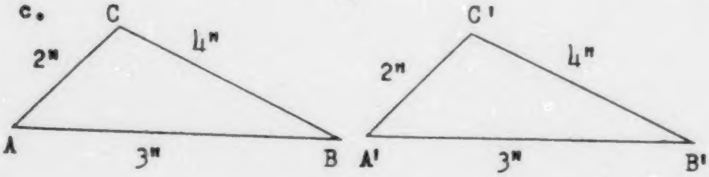
2. Points P, Q, and R are located on the sides AB, BC and CA respectively of $\triangle ABC$. If line segments PQ, QR and RP are drawn, and if $PQ = \frac{1}{2}CA$, $QR = \frac{1}{2}AB$, and $RP = \frac{1}{2}BC$, prove that $\angle PQR = \angle A$.

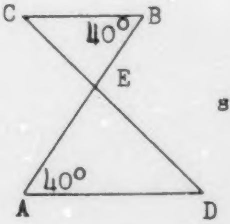
3. State the theorem which proves the triangles similar in each case:





PT,QS are straight lines





AB,CD are straight lines

4. The special right triangle whose sides are in the ratio 3:4:5 should be considered here. Why are all 3-4-5 triangles right triangles? If a right triangle has legs of 6" and 8", how long is the hypotenuse? Why?

UNIT TEST

SIMILARITY

1. Name 2 sets of quadrilaterals such that any member of one set would have its corresponding angles equal to those of any member of the second set, but the two are not similar.

2. Same as above. Require that the corresponding sides be in proportion but the two quadrilaterals are not similar.

3. In triangle ABC, a line parallel to AC cuts AB at D and BC at E. If AD is 4, DB is 8 and BC is 18, find EC.

4. The bases of a trapezoid are 8 and 12 and the altitude is 3. If the non-parallel sides are extended until they meet, find the altitude of the triangle formed by them and the smaller base of the trapezoid.

5. Construct an equilateral triangle given its perimeter.

6. Show that A(-2,1), B(3,5), C(4,9), and D(-1,5) are the vertices of a parallelogram.

7. Line segment RS joins two points R and S which lie on opposite sides of parallelogram ABCD. RS cuts diagonal AC in point T. The product of RT and a certain one of the segments into which the diagonal is cut by T equals the product of ST and the other segment of the diagonal. Write the correct equal products and prove that they are equal.

8. Prove that two triangles are similar if they agree in two angles.

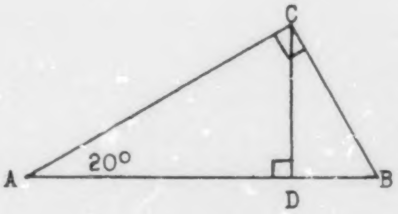
IX.

MEAN PROPORTIONAL AND PYTHAGOREAN THEOREM

LESSON 1

Aim: to investigate the similar triangle relationships in a right triangle which has an altitude to its hypotenuse.

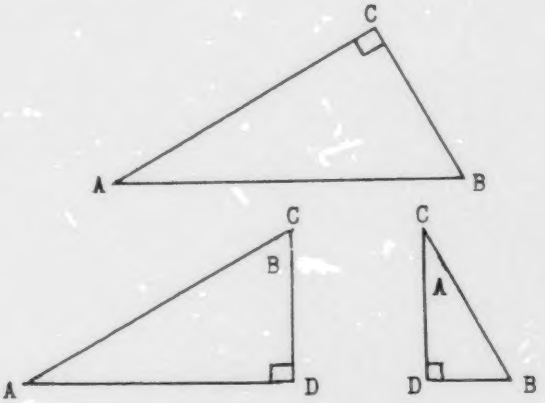
Development:



Have the students find all the other angles in the figure at the right if $\angle A$ is 20°. Change the number of degrees in $\angle A$. Class will note that $\angle ACD$ is equal to $\angle B$ and $\angle DCB$ is equal to $\angle A$ for all values of $\angle A$ less than 90°.

Have the students deduce (orally) that the two small triangles are each similar to the large one and also similar to each other.

Use three separate diagrams for the original triangle and the two smaller triangles. Also, have available three paper triangles related in the same way.



Have the students list possible proportions for each pair of triangles and state these propositions. Two propositions which should be deduced are:

> If an altitude is drawn to the hypotenuse of a right triangle, the altitude is the mean proportional between the segments of the hypotenuse.
>
> If an altitude is drawn to the hypotenuse of a right triangle, either arm is the mean proportional between the hypotenuse and the projection of that arm on the hypotenuse.

Note: The concept of a projection may be vitalized by discussing the function of a slide or movie projector.

When these theorems are applied, some student weaknesses in algebra will become apparent. This is the time to review the algebra. For example, if $x^2 = 36$, why does $x = 6$? (Positive square roots of equals are equal.)

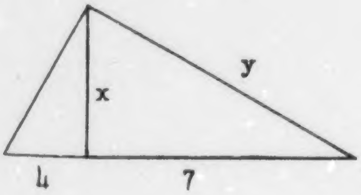
Suggested exercises may be found in any good textbook.

LESSON 2

Aim: to clarify the distinction between rational and irrational numbers as a result of the application of the right triangle with an altitude to the hypotenuse theorem

Development:

Have the students find the value of x and y in the figure at the right.



The resulting equation $x^2 = 28$ and $y^2 = 77$ have irrational solutions: $x = \sqrt{28}$ and $y = \sqrt{77}$.

Here is an opportunity to reinforce and extend the student's concept of irrational numbers. The teacher may review the idea of integers by using a number line. The students should be led to realize that despite the fact that man invented names for an infinite number of integers to be used in counting, he needed a new type of number when he tried to find the measure of certain line segments (see above)

Have the students learn that a rational number is one which can be put in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$. They should realize that the number $\sqrt{28}$ which occurred in the lesson is not a rational number.

Note: For enrichment, show $\sqrt{2}$ is irrational.

In the table of square roots $\sqrt{28}$ is listed as 5.29. However, the student should realize that it is not exactly 5.29 (which is $\frac{529}{100}$ and therefore rational) but 5.29... to an infinite number of decimal places. Encourage the student to estimate the square roots before looking up their decimal values in the table.

Initially, at least, the answer should not be left in radical form. For example, $\sqrt{3}$ is meaningless to many students as evidenced by the common error of thinking a number such as $7\sqrt{3}$ is a number between 7 and 8.

The students learned a method for finding the square root of a number in Grade 9. Simplification of radicals may be postponed until after the Pythagorean Theorem is developed and the 30°, 60°; 45° relationships are discussed.

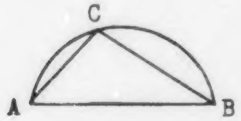
Suggested exercises may be found in any good textbook.

LESSON 3

Aim: to apply the "altitude mean" theorem to an application involving "an angle inscribed in a semicircle"

Development:

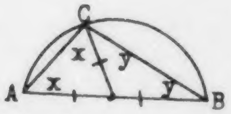
Experiment: Have the students draw a semicircle with diameter AB and any $\angle ACB$ with point C on the circle. What do you discover?



The students will recall that $\angle ACB$ is a right angle.

Have the students consider an algebraic proof that $\angle C$ is a right angle.

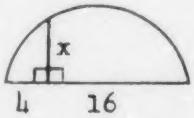
$$2x + 2y = 180^\circ$$
$$x + y = 90^\circ$$



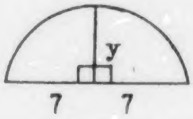
Note: This is a good example of the greater use of algebra in the syllabus.

Suggested exercises:

1. Find x.

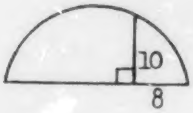


2. Find y.



The arcs shown are semicircles.

3. A truck 10' high is being driven through a semicircular tunnel. The truck is 8' from the right side of the tunnel. What is the diameter of the tunnel if the truck just fits?



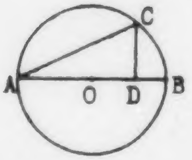
Lesson 4

Aim: To deduce line segments in a circle in proportion using similar triangles, and to use these situations to introduce quadratic equations of the form $ax^2 + bx + c = 0$, where $b \neq 0$ and $c \neq 0$.

Development:

Have the students review the methods of deducing line segments proportional. Have them also review how to select those triangles which are to be shown similar in order to deduce a particular proportion (see lessons 4,5,6 of Chapter VIII).

Suggested exercises:



1. Given: Circle O with diameter AB, chord AC, $CD \perp AB$

   Deduce: $\frac{AC}{CB} = \frac{AD}{CD}$

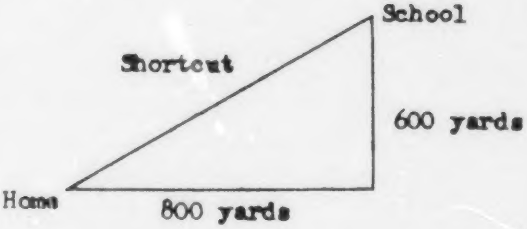
2. In exercise 1, if AC is 4" longer than CD, CB = 3", and AD = 7", find CD.

Note: To solve the resulting quadratic equation $x^2 + 4x = 21$, it may be necessary to review factoring of a trinomial and "If a product equals zero, one of the factors is zero." The solution of the complete quadratic is limited to the method of factoring.

LESSONS 5 and 6

Aim: to deduce the Pythagorean Theorem

Development:



Experiment: Find the length of the shortcut from home to school.

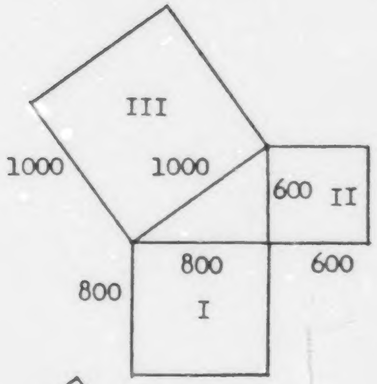
Some of the students may recall the Pythagorean Relation from their work in Ninth Year Mathematics. Ask for an estimate of the length of the shortcut. An intelligent estimate would be more than 800 yards but less than 1400 yards.

It may be suggested that an altitude be drawn to the hypotenuse. If so, the segments should be called x and c-x rather than x and y. The relations:

$$\frac{c}{600} = \frac{600}{x} \qquad \text{and} \qquad \frac{c}{800} = \frac{800}{c-x}$$

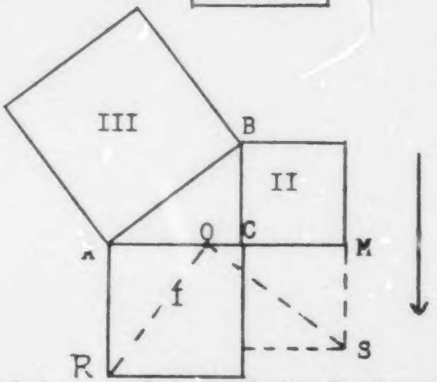
lead to:

$$c^2 = 600^2 + 800^2$$
$$c^2 = 360000 + 640000$$
$$c^2 = 1000000$$
$$c = 1000$$



To clarify the significance of $1000^2 = 600^2 + 800^2$, the students might examine the diagram at the right. They should be led to understand that the first and second squares actually cover exactly as many square units as does the third square.

Some students are inclined to doubt that the first and second squares can actually be made to cover the third square. One interesting way to show this is as follows:



Cut out a right triangle and three squares as shown at the right. (Use a different color for each square.) Move square II next to square I so that ACM is a straight line. Find O so that MO = AC. Cut along OR and OS. This will result in five pieces from square I and II. Rearrange these pieces on top of square III, placing ROS on one corner of the big square.

The students should be made aware of some of the history and uses of the Pythagorean Theorem.

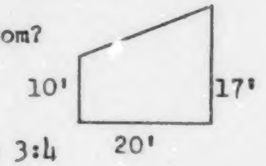
Deduce the theorem using the original experiment as a model.

Suggested exercises:

1. Find the distance from home to second base of a baseball diamond.

2. Will a 28" umbrella fit in a 18" by 24" suitcase?

3. How long is the sloping beam in this attic room?



4. The arms of a right triangle are in the ratio 3:4 and the hypotenuse is 45. Find the two arms.

5. The sides of a certain right triangle may be represented by three consecutive integers. Find them.

LESSONS 7 and 8

Aim: to apply the Pythagorean Theorem to the distance between two points in coordinate geometry

Development:

Have the students find the distance between (3,7) and (-5,7); the distance between (3,7) and (11,13).

The distance between (3,7) and (11,13) will lead to the use of the Pythagorean Theorem to deduce

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} \quad \text{or} \quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note: A quadrilateral can now be shown to be a parallelogram by showing two pairs of opposite sides equal. However, it is still easier to show that the diagonals have the same midpoint.

Students may need to review radicals.

Exercises follow in which the distance formula is essential. Naturally, not all these types of exercises should be done at once, but the topic should be kept alive through spiral homework assignments.

1. Plot A (-3,2), B (5,-2), C (10,10). Find the length of MN, the line joining the midpoints of AC and BC. Compare the length of MN with the length of AB. This is an illustration of what theorem?

2. Plot A (-3,0), B (1,-2), C (5,6). Show that ABC is a right triangle.

Note: This may be the student's introduction to the converse of the Pythagorean Theorem. If so, he should of his own accord comment on the need for proving the converse. The teacher should go through the deductive proof informally. The converse of the Pythagorean Theorem may be proved analytically. See Lesson 11 of this chapter.

Acceptable student answer:

$$AB = \sqrt{(4)^2 + (-2)^2} = \sqrt{16 + 4} = \sqrt{20}$$

$$BC = \sqrt{(4)^2 + (8)^2} = \sqrt{16 + 64} = \sqrt{80}$$

$$AC = \sqrt{(8)^2 + (6)^2} = \sqrt{64 + 36} = \sqrt{100}$$

Is $(AC)^2 = (BC)^2 + (AB)^2$?

$$(\sqrt{100})^2 \overset{?}{=} (\sqrt{80})^2 + (\sqrt{20})^2$$

$$100 \overset{?}{=} 80 + 20$$

Yes. Therefore, triangle ABC is a right triangle.

3. Find the perimeter of the triangle whose vertices are (-1,-2), (3,5) and (0,1).

4. Find the length of the median to side AC of the triangle formed by A (-2,4), B (3,6), C (6,-2).

5. The center of a circle is at (4,5) and it passes through (9,5). Will it also pass through (7,9)?

Note: The student should take pride in completing this exercise without resorting to squared paper.

6. The following exercises may be used as enrichment:

a. Plot A (-1,-1), B (1,3), C (2,5). Show that they are collinear.

Student reviews, 'Sum of two sides of a triangle is greater than the third,' and hopes that AB + BC = AC

$AB = \sqrt{(2)^2 + (4)^2} = \sqrt{20}$

$BC = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$

$AC = \sqrt{(3)^2 + (6)^2} = \sqrt{45}$

Is $\sqrt{20} + \sqrt{8} = \sqrt{45}$?

$2\sqrt{5} + \sqrt{5} \overset{?}{=} 3\sqrt{5}$

Yes. Therefore, the points are collinear.

Note: Student reviews idea of radicals and is introduced to addition of radicals.
This type of exercise can be done through the use of slope.

b. Show that the median to the hypotenuse of any right triangle is half the hypotenuse.

The student who completes this on his own has acquired a great deal of power with an important tool of mathematics. He should suggest A (0,0), B (a,0), C (0,b) as general vertices.

c. A different type of example in which the coordinate grid serves a useful purpose as an aid to student comprehension is, 'Show that the irrational number $\sqrt{5}$ has a length which is between 2 and 3.'

Discuss the infinitude of rational numbers between 2 and 3, and point out that $\sqrt{5}$ is not one of them. Students should review that a rational number is one which can be written in the form $\frac{p}{q}$ where p and q are integers. $\sqrt{a}$, with 'a' not a perfect square, can be shown in higher mathematics to be irrational. They will enjoy the thought that there is an even higher order of infinitude of irrational numbers than rational numbers.

Select points (0,0) and (2,1) and use compasses to lay off the hypotenuse on the X-Axis so that the real but incommensurable length of $\sqrt{5}$ becomes visual. Repeat for other irrational numbers.

LESSON 9

Aim: to note certain relationships existing in a 30°, 60°, 90° right triangle
to reinforce and extend the students' understanding of irrational numbers

Development:

Experiment: Construct an equilateral triangle with a 2" side. Construct an altitude. How long is the altitude?

Some student answers will be 1 3/4, 1 5/8, 1.75, or 1.625. Have the students realize that these are rational numbers which are merely convenient estimates for the length of the altitude.

Can we find the length of the altitude in another way? The use of the Pythagorean Theorem will yield $2^2 = 1^1 + x^2$. From this, the length of the altitude is $\sqrt{3}$ or an irrational number. $\sqrt{3} = 1.732$ only when rounded to the nearest thousandth.

Have the students use the Pythagorean Theorem to find the altitude of an equilateral triangle whose side is 5. It will be necessary to review at this point the simplification of radicals.

Have the students now find the altitude of an equilateral triangle whose side is s. Have them reach the generalizations concerning the ratio of the side opposite the 30° angle and the hypotenuse, and the side opposite the 60° angle and the hypotenuse. The students should realize that all 30°, 60°, 90° right triangles are similar and the ratios are therefore constant.

Note the frequent use by draftsmen of the right triangle which is half an equilateral triangle.

Have the students now do exercises involving the 30°, 60°, 90° right triangles. Have them frequently estimate the value of such numbers as $7\sqrt{3}$.

LESSON 10

Aim: to discover the relationship in an isosceles right triangle
to reinforce and extend the student's understanding of irrational numbers

Development:

In a manner similar to that in lesson 9, have the student generalize how to find the length of a diagonal of a square when the length of a side is given, and how to find the length of a side when the length of the diagonal is given.

Have the students draw on square-ruled paper a right triangle whose legs are each 1 unit. They should be led to understand that the hypotenuse, whose length is $\sqrt{2}$, can be laid off on the X-Axis and will fall between $x = 1$ and $x = 2$. This serves to reinforce the idea that in addition to the infinitude of rational numbers between any two integers, there exists many irrational numbers.

Have the students now do exercises involving the 45°, 45°, 90° right triangle.

LESSON 11

Aim: to deduce and apply the converse of the Pythagorean Theorem

Development:

Have the students try to show that triangle ABC is a right triangle if its vertices are A (3,3), B (14,1), C (11,7).

Some students will suggest that since the sum of the squares of the two sides equals the square of the third side, the triangle is a right triangle. Lead the class to realize that these students have assumed the converse of the Pythagorean Theorem. The students will now see a need to deduce the converse.

Given: triangle ABC and $c^2 = a^2 + b^2$

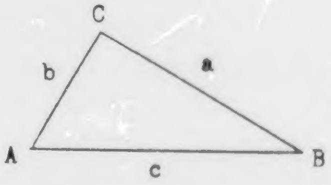
Deduce: triangle ABC is a right triangle with right angle at C

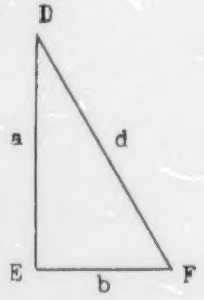
Construct right triangle DEF with legs a and b.

$$a^2 + b^2 = d^2$$
$$a^2 + b^2 = c^2$$
$$d^2 = c^2$$
$$d = c$$

$\triangle DEF \cong \triangle ABC$ (SSS = SSS)

$\angle C = \angle D$ = right angle



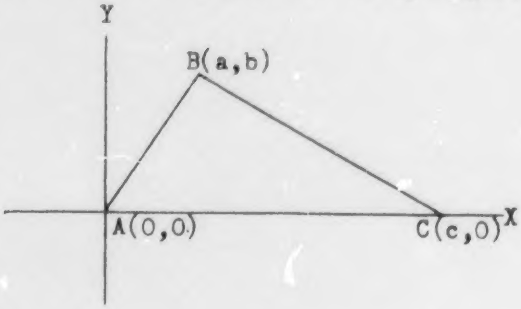


Students will enjoy the following deduction of the converse of the Pythagorean Theorem using coordinate geometry:

Given: Any $\triangle ABC$.

Select the X-Axis so that it passes through points A and C. Select the Y-Axis so that it passes through point A.

Then A(0,0), B(a,b), C(c,0) are the vertices of a triangle such that $AB^2 + AC^2 = BC^2$.



Prove: ABC is a right triangle.

By the distance formula, the given fact becomes

$$\left(\sqrt{a^2 + b^2}\right)^2 + c^2 = \left(\sqrt{(c-a)^2 + (-b)^2}\right)^2$$

Therefore $a^2 + b^2 + c^2 = c^2 - 2ac + a^2 + b^2$

or $0 = -2ac$

But c is not zero because A and C are distinct points. Therefore a must be zero, Point B is therefore (0,b) and lies on the Y-Axis. Hence BAC is a right angle.

Note: The method of showing perpendicularity by slopes is not in the syllabus, even among the optional topics. Some classes probably could do this proof using slopes.

Have the students now turn their attention to the exercise given at the beginning of the period. Note how much is written as an answer.

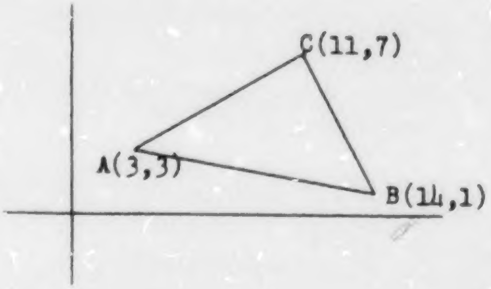
Acceptable student response:

$(AC)^2 = 8^2 + 4^2 = 80$

$(BC)^2 = 3^2 + (-6)^2 = 45$

$(AB)^2 = 11^2 + (-2)^2 = 125$

$80 + 45 = 125$

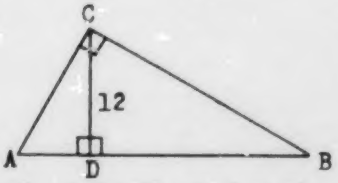


Therefore, triangle ABC is a right triangle because, "If the sum of the squares of two sides of a triangle is equal to the square of the third side, it is a right triangle.

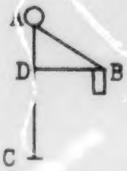
Have students do exercises involving the converse of the Pythagorean Theorem.

UNIT TEST

MEAN PROPORTIONAL and PYTHAGOREAN THEOREM



1. State and prove the Pythagorean Theorem.
2. DB exceeds AD by 7. Find AD.
3. Given quadrilateral ABCD. A (-2,4), B (3,7), C (2,5), D (-3,2).
   a. Show that ABCD is a parallelogram.
   b. Is ABCD a rectangle? Why?
4. Which of the following is irrational:
   a. .6666...
   b. $\sqrt{25}$
   c. $\sqrt{7}$
   d. 2.4



5. AC is a 12' lamppost and at B there is a traffic light. If B is 8' from the ground and arm BD is 5' long, how long a brace is needed for AB?
6. Honor: Derive the equation of the locus of all points equidistant from (-5,7) and (9,7).

X.

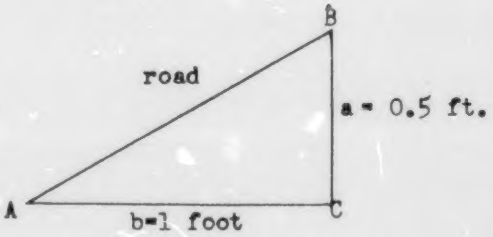
TRIGONOMETRY AND SLOPE

LESSON 1

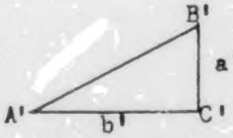
Aim: to develop and apply the concept of the tangent of an acute angle

Development:

Pose the problem of determining the "grade" or "slope" of a road in relation to the "angle of steepness." For example, a straight road rises 0.5 of a foot for every foot of horizontal distance. What is the angle at which the road rises from the horizontal? Have the class make a scale drawing and find the angle to the nearest degree by means of a protractor.



Have the class understand that the ratio of "rise" to "run" in the scale drawing is the same as in the actual physical setting, because of the proportion obtained from similar right triangles: $a/b = a'/b'$ for a fixed angle A.



Scale: 1"=1'

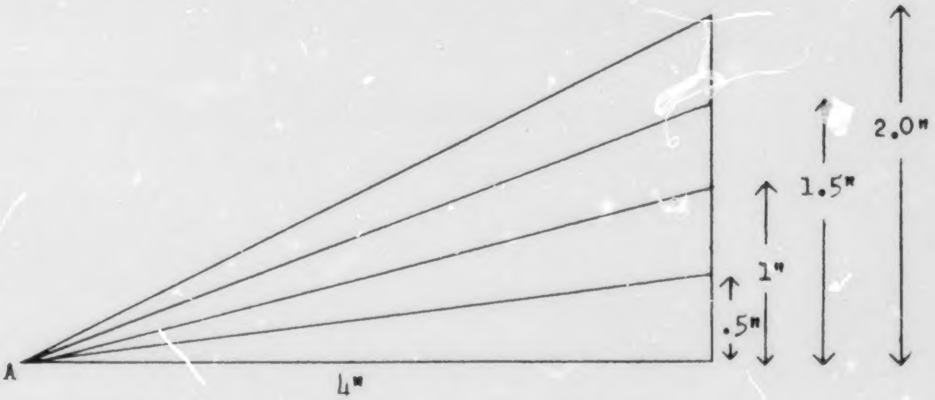
Note: It will be necessary to point out that the ratios are formed first in one triangle, then in the other triangle, instead of from one triangle to the other triangle, as the student has been accustomed to doing. This is justified by "alternation" of the terms of the proportion, so that $a/a' = b/b'$ is equivalent to $a/b = a'/b'$.

Emphasize that using a different scale leads to the same angle.

Have the class do scale drawing experiments for different "rises" in the same "run" and build a table of ratios of "side opposite acute angle to side adjacent to acute angle" corresponding to the "degrees in angle". Introduce the definition "tangent of angle," and the notation "tan A."

Result of Experiment

tan A	∠A
.125	7°
.250	14°
.375	21°
.500	27°



Have the class deduce from similarity of triangles, "If two right triangles agree in an acute angle, they agree in the ratio of the leg opposite the acute angle to the leg adjacent to the acute angle." In each case, there is an "ordered pair" of numbers, the "number of degrees in the angle" and the "tangent of the angle," in 1-1 correspondence, so that if the first of the two numbers is known, the other is determined."

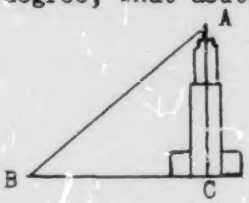
Introduce the "table of tangents" as a list of such ordered pairs. Inform the class that this table was worked out, not by experiment, but by the use of a formula taken from advanced mathematics (calculus).

Have the students practice using the table; e.g., "If $\angle A = 70^\circ$, find tan A"; "If tan x = .78, find $\angle x$ to the nearest degree."

Have the students use the table of tangents to solve problems such as the following:

1. The wire from the top of a telephone pole 20 feet high reaches the ground 10 feet from the foot of the pole. To the nearest degree, what acute angle does the wire make with the ground?

2. To find the height (AC) of the Empire State Building, a point (B) was located 920 feet from the foot (C) of the building, and $\angle CBA$ was found to be 58°. Find the height to the nearest foot.



The assignment should provide for both practice in using the tangent table and for additional problems involving the tangent ratio.

LESSON 2

Aim: to discover and use the sine and cosine relationships in the right triangle

Development:

Motivate the work with a problem such as: "A recommended safety angle for a ladder is an angle of 75° with the ground. How high above the ground will a 20-foot ladder reach if placed at this angle?" Have the student realize that a ratio other than the tangent is needed to solve the problem.

Introduce the definitions and notation for the sine and cosine of an angle. Have the class deduce, by using similar triangle theorems, that for two right triangles $\angle A = \angle A'$ implies that sin A = sin A', and conversely. Do the same for the cosine ratio.

Solve the motivation problem, and a similar problem involving the cosine ratio, using the table of ratios. In further class problems, have the class solve for the angle when the hypotenuse and the opposite leg or adjacent leg are given.

Note: The problem of finding the hypotenuse, given the angle and one of the legs, will present two difficulties: (a) solving a fractional equation or a proportion, and (b) using long division. Point out that the long division cannot be avoided by taking the complementary angle, as it can be avoided in the similar case with the tangent ratio.

In further class practice and in the assignment, include both types of sine problems, both types of cosine problems, and the previous two types of tangent problems.

LESSONS 3 and 4

Aim: to make further applications of the trigonometric ratios to

problems involving angle of elevation and angle of depression
finding the altitude of a triangle or parallelogram
finding a distance requiring the use of two right triangles

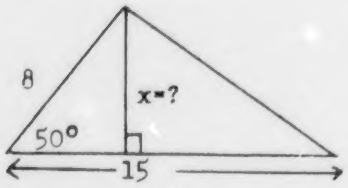
Development:

A model, homemade transit (protractor on yardstick), or a "clinometer" will help clarify the concept of "angle of depression (elevation). Re-emphasize the importance of the right triangle and the trigonometric ratios in practical work, as in navigation, surveying, and astronomy, (indirect measurement).

Problems to be presented for solution will include the following types, which are to be found in standard textbooks:

1. Find the height of a cloudbank if its angle of elevation is 86° at a point on the ground 800 feet from a point directly beneath the cloudbank.
2. How far is a ship from a vertical cliff 200 feet high, if from the top of the cliff the angle of depression of the ship is 20°?

3.



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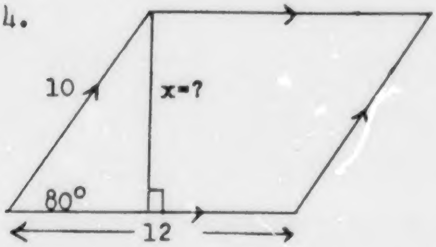

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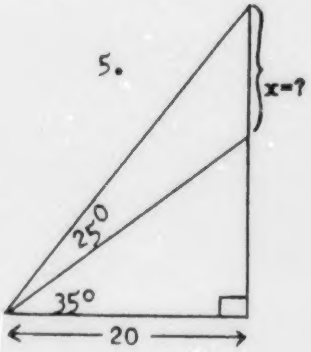


Enrichment: A triangle (or parallelogram) has two given sides. How does the altitude to one of those sides vary as the angle between the given sides varies?

In the review and further application, supplement the numerical work with exercises in which the student is to determine (from the given parts, and part to be found) the particular ratio (sine, cosine, tangent) to be used.

UNIT TEST

NUMERICAL TRIGONOMETRY

1. Using the table, find each ratio:

   a. tan 68° b. sin 40° c. cos 75°

2. Using the table, find each angle x: (to nearest degree)

   a. $\tan x = 3/5$ b. $\sin x = .9703$ c. $\sin x = .4800$
   d. $\cos x = .4695$

3. Find the height of a tree which casts a shadow 12 feet long when the sun's rays make an angle of 70° with the ground.

4. At the instant when a tree fifty feet high casts a shadow forty feet long, what angle do the sun's rays make with the ground?

5. How high is a kite if the 200 foot string to which it is attached makes an angle of 48° with the ground? (Assume that the string is straight.)

6. A mine tunnel is straight and slopes downward so that when a miner goes 400 feet into the tunnel he is 120 feet below the surface. At what angle does the tunnel slope?

7. A 20-foot ladder reaches a point on a wall 18 feet from the ground. What angle does the ladder make with the wall?

8. The sides of a triangle are 20 and 30 and form an angle of 40°. Find the altitude of the triangle drawn to the longer of the two given sides.

9. In an isosceles triangle the equal sides are each 12 and the base is 10. Find the number of degrees in each base angle.

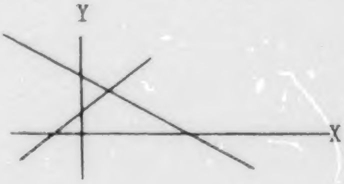
10. In $\triangle ABC$, CD is the altitude to side AB. If $\angle ACD = 30^\circ$, $CD = 20$, and $\angle BCD = 50^\circ$, find the length of side AB.

LESSONS 5, 6 and 7

Aim: to define the slope of a line segment
to deduce that every line segment of a straight line has the same slope
to define the slope of a line as the slope of any of its line segments
to develop that in a set of points, if the slope is the same for every line segment joining the points in pairs, then the set of points lies on a straight line.(are collinear)
to deduce that if two straight lines are parallel, they have the same slope, and conversely
to deduce that if two straight lines are perpendicular, the product of their slopes is -1, and conversely.

Note: The topic of slope is now optional in the Tenth Year Mathematics Syllabus. Exercises on the Tenth Year Regents which involve slope have tended to be very elementary.

Development:



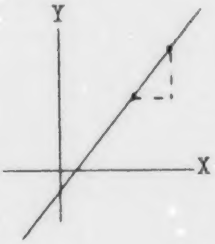
Challenge the students to describe "how much" these sloping lines (not parallel to the axes) do slope.

Develop the definition: The slope of a line segment joining two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is the ratio $\Delta y / \Delta x$ where $\Delta x \neq 0$ and $\Delta y = y_2 - y_1$, $\Delta x = x_2 - x_1$.

Have the students deduce that every line segment of a straight line has the same slope.

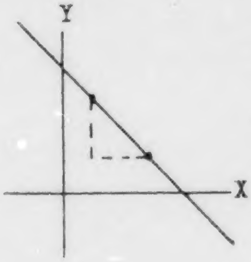
They should now define: The slope of a line is the slope of any of its line segments.

Discuss the need for positive and negative slope. Use summary as follows:



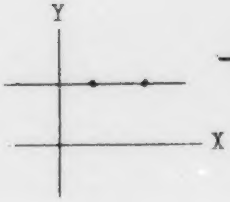
$\frac{\Delta y}{\Delta x}$ is positive

Slope is positive



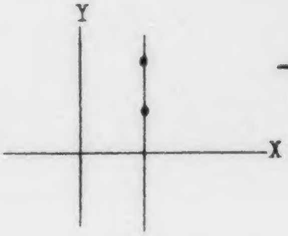
$\frac{\Delta y}{\Delta x}$ is negative

Slope is negative



$\frac{\Delta y}{\Delta x}$ is zero

Slope is zero



$\frac{\Delta y}{\Delta x}$ does not exist

(We cannot divide by zero)

Have the students see that "constancy of slope" among three points implies that the points are collinear.

Have the students deduce that if two straight lines are parallel, they have the same slope. A brief outline of the deduction follows:

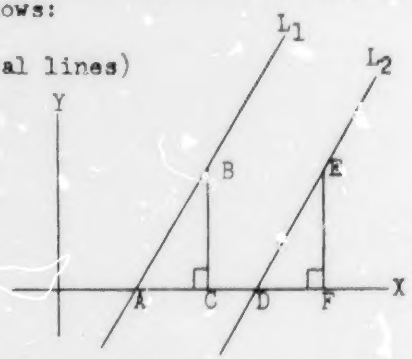
Given: $L_1 \parallel L_2$ (L_1 and L_2 are non-vertical lines)

Deduce: Slope of L_1 = slope of L_2

Slope of L_1 = CB/AC

= FE/DF ($\triangle ABC \sim \triangle DEF$)

= Slope of L_2



Conversely, it can be shown that if the slopes of two straight lines are equal, the triangles are similar and L_1 and L_2 are parallel.

Review two ways of showing that a quadrilateral is a parallelogram - both pairs of sides are parallel (definition) or that one pair of sides is equal and parallel (theorem). "Parallel" is equivalent to "have same slope."

Have the students deduce that if two straight lines are perpendicular, the product of their slopes is - 1. A brief outline of a suggested deduction follows:

Given: $L_1 \perp L_2$ (Where L_1 and L_2 are both non-vertical)

Deduce: $m_1m_2 = -1$

Place a set of coordinate axes on the figure formed by L_1 and L_2 so that their intersection, C, is on the y-axis. Call the point where the x-axis cuts L_1 the point A (-a,0), and the point where the x-axis cuts L_2 the point B (b,0). Note: a and b are positive. The coordinates of C will be (0,y).

Since $\angle C$ is a right angle, CO is the altitude to the hypotenuse of right triangle ABC.

Therefore, $\dfrac{a}{y} = \dfrac{y}{b}$

$y^2 = ab$

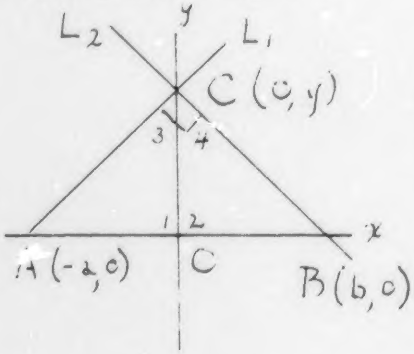
The slope of $L_1 = m_1 = \dfrac{y - 0}{0 - (-a)} = \dfrac{y}{a}$

The slope of $L_2 = m_2 = \dfrac{0 - y}{b - 0} = \dfrac{-y}{b}$

$m_1m_2 = \dfrac{-y^2}{ab}$

But $y^2 = ab$

Therefore, $m_1m_2 = -1$



Now suppose we are given that the product of the slopes of two lines is -1.

Using the diagram above, $m_1 = \frac{y}{a}$ and $m_2 = \frac{-y}{b}$

$$\text{Since } m_1 m_2 = -1, \ \frac{y}{a}\left(\frac{-y}{b}\right) = -1$$

$$y^2 = ab \text{ and therefore } \frac{a}{y} = \frac{y}{b}$$

Since right angles 1 and 2 are equal and the including sides are in proportion, △ AOC ~ △ COB.

Therefore, ∠A = ∠4

But ∠3 is the complement of ∠A

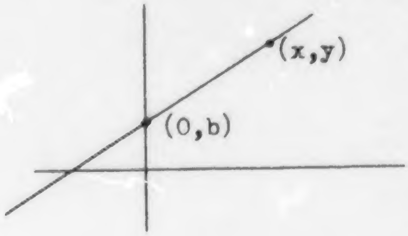
Therefore, ∠3 is the complement of ∠4

Therefore, ∠ACB is a right angle.

We have thus proved that, "If the product of the slopes of two lines is -1, then the lines are perpendicular." (Incidentally, we have also proved that, "If an altitude to the base of a triangle is the mean proportional between the segments of the base, then the triangle is a right triangle and the base is the hypotenuse.")

If the optional topic concerned with equations of lines is taught, develop the theorem that the equation of a line whose slope is m and whose y-intercept is b is

$$y = mx + b$$



$$m = \frac{y - b}{x - 0}$$

$$mx = y - b$$

$$mx + b = y$$

Suggested exercises:

1. Plot the points A (3,1), B (-4, -2), and C (10,4) and show that they are collinear. Use slopes.

2. Show that the line joining the midpoints of any two sides of a triangle is parallel to the third side. Use (0,0), (a,b), (c,d) as the vertices of the triangle.

3. Show that:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

4. Find the number of degrees in the angle formed by a line containing (2,3) and (-1,-4) and the X-Axis (Y-Axis). Use trigonometry of the right triangle.

5. Show that the diagonals of a square are perpendicular by means of slopes. Use (0,0), (a,0), (a,b) and (0,b) as vertices of the square.

6. The vertices of a triangle are A (2,3), B (27, 3), and C (11,15). Show by means of slopes that the triangle is a right triangle.

Note: The syllabus suggests several specific types of exercises under equations of lines (optional). See page 32, note 22 of the syllabus.

UNIT TEST

TRIGONOMETRY and SLOPE

1. Plot the points A (-2,-3), B (1,1), and C (4,5).

   a. Find the slopes of the line segments AB and BC.
   b. Comparing the results, what conclusion can be drawn concerning the three given points?

2. Plot the points A (0,0), B (4,2), C (2,-2), and D (-2,-4).

   a. By means of slopes, show that ABCD is a parallelogram.
   b. Find the slopes of the diagonals of ABCD.
   c. How are the diagonals related? What kind of parallelogram is ABCD?

3. Show by use of the slope formula that the points

   P (3,5), Q (5,7), and R (6,2) are the vertices of a right triangle.

4. Find to the nearest degree the acute angle between the X-Axis and the line $y = \frac{1}{2}x + 3$.

XI.

AREAS

LESSON 1

Aim: to introduce area and the need for a standard unit of area

Development:

Have the students suggest the need for measuring the extent of flat surfaces that are bounded by squares, rectangles, parallelograms, and so on. Mention should also be made of the need for measuring areas on curved surfaces (in calculating metabolism rates which depend partly on skin areas). Other examples are the areas of airplane wings, areas of floors in buying carpets, areas of surfaces to be painted in estimating painting costs, and so on.

There is also a theoretic value of area measures in calculating other quantities such as volumes, densities, moments, and center of gravity.

Note: This discussion should tap whatever information is possessed by students. The teacher should not hesitate to furnish information where needed. The discussion might take at least a half period. It may be anecdotal such as the method of covering an irregular surface with tin foil and weighing the tin foil to deduce the area.

The teacher can now have the students consider area as a numerical measurement. A general discussion of the nature of measuring units and what is required of them to be satisfactory should contain the following points:

> The unit of area should itself be an area and hence should be in the form of a closed plane figure.
>
> The unit should be convenient. Since we have a "rectangle civilization," that is, many of the objects we find in our culture have the shape of rectangles, the unit area figure should have right angles. Another convenience is found in the property that four right angles cover all the space about a point.
>
> The unit should be standardized. This leads us to measure the sides of our unit in linear units that are already standardized. This, in turn, leads to choosing a square because of the equality of its sides.

At this point, it is well to distinguish between a square that is an area unit, to be called a unit square, and an area that measures one square unit. This distinction becomes clearer in contrasting the area of a rectangle (2" by 1") that has 2 square inches with a 2 inch square that has 4 square inches. It is also important to note that a square inch may be the area measure of a figure that is not an inch square. This may be done by cutting an enlarged unit square along an axis of symmetry and piecing the parts together to form either a rectangle (2" by $\frac{1}{2}$") or an isosceles triangle.

The cut-out figures can well be moved about on the ground glass table of an overhead projector.

It would be a good exercise for students to form a number of figures having different shapes but each having an area 4 square inches. Are the perimeters of these figures equal?

LESSON 2

Aim: to show the relationship between congruence and equality of area
to discover and verify the area formulas for the rectangle, square, parallelogram, triangle, and trapezoid by experimentation

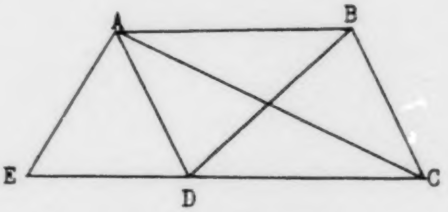
Development:

Have the students discover the relationship between congruence and equality of areas, as well as the symbolism of congruence, similarity, and equality. Knowing that congruence is a sufficient condition for equality, the class may now consider such propositions as, "The diagonal of a parallelogram bisects the area of the parallelogram" or "The bisector of the vertex angle of an isosceles triangle bisects the triangle." The phrase, "bisects the triangle," should be defined.

A triangle (or any other figure) is bisected if it is divided into two figures having equal areas. We may refer to the equality of any two plane figures (such as a triangle and rectangle) if their areas are equal. Contrast this sense of equality with that of similarity and congruence.

Have the class now apply the equality axioms (identity, equals plus (or minus) equals...) to such exercises as:

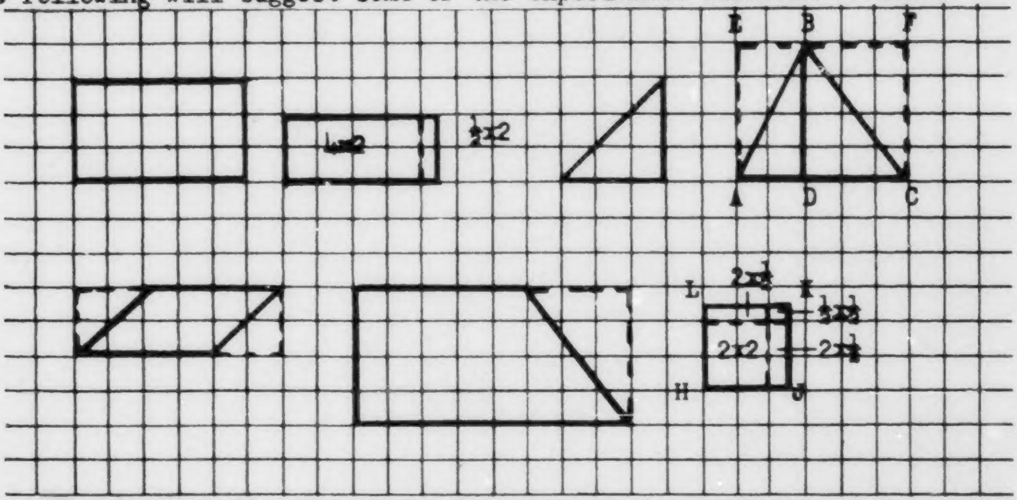
$\triangle AEC + \triangle ACB = ?$
$\triangle AEC - \triangle AED = ?$
$ABCD + ADE = ?$
$ABCE - AED - DAC = ?$



Present the class with a rexographed sheet showing various common figures on squared paper, as shown below. On the overhead projector, a set of the same figures drawn on acetate and another separate sheet of square-ruled acetate will permit placing the squares on the figures in various positions. This should be done quickly to permit rediscovery of the areas of the rectangle, square, parallelogram, triangle, and trapezoid as "inductive propositions."

The experiments should include cases of rectangles and squares whose dimensions are an integral number of units or fractional units, particularly halves. In like manner, the verification of the parallelogram, triangle, and trapezoid formulas should be conducted.

The following will suggest some of the experiments referred to above:



Note: The fractional parts of square units may be easily counted by noting that the diagonal of a rectangle bisects it. See the triangle above in which ABD is half EBDA (½x8) and BDC is one half of BFCD (½x12). Therefore, ABC = 4+6 = 10 sq. in. On the other hand, ½bh = ½ x 4 x 5 which is also 10 sq. in. The square HJKL furnishes a verification of the product, 2½ x 2½ = 4 + 2x½ + ½x½ = 6¼.

T: Can the number of units in area of any rectangle, square, etc. be counted in this way? Have we made any deliberate restriction in the choice of lengths for the sides of our figures?

This is an opportunity to increase the maturity of students' understanding of incommensurable line segments. Tell them that up to now in this lesson they have been assuming that the length of a line segment is always rational, that is, can be expressed as a ratio of two integers, like 6¼, which is $\frac{25}{4}$. Recall that the lengths might be irrational numbers like $\sqrt{5}$.

Students may think that taking a small enough unit will permit us to express the dimensions of any figure as integral multiples of that unit. However, this is not always possible. In such a case we say the two lengths are incommensurable. For example, if the side of a square is 1", its diagonal is $\sqrt{2}$"; the side and diagonal are incommensurable. We could draw a rectangle whose length is $\sqrt{2}$" and whose width is 1". Even if both dimensions are irrational, such as $\sqrt{2}$ and $\sqrt{3}$, it is not always possible to find a unit to permit us to express each of the dimensions in terms of it.

Therefore, if our theory of areas is to cover every case, including the incommensurable, we must base it on at least one proposition which is postulated. Ask the class whether they should postulate all five inductive propositions. Did it seem as if, in our counting, any one of them might have been used as a basis for deriving all the others? Guide the students into selecting and listing the postulate, "The area of a rectangle is the product of its length and width."

Note to teacher: The same difficulty arises in the postulate, "The line parallel to one side of a triangle divides the other two sides proportionally." If area is taught before similarity, that postulate would be deducible from the area sequence, thus reducing two complicated assumptions to one. S.M.S.G. has proposed this approach.

Lessons 3, 4 and 5

Aim: to develop an appreciation of the explanatory values inherent in a postulational system
to exhibit the value of deductive reasoning in discovering new theorems

Development:

Accepting the five area formulas as the result of experimental verification, the class is now ready to consider the problem of arranging them in logical order.

The following interchange between teacher and students illustrates how these five propositions may be arranged in a logical order, that is, in such an order that one may be deduced from preceding ones.

Teacher: Let us consider squares and rectangles. Which is the superset and which is the subset?
Student: Squares form a subset of rectangles.
Teacher: Will formulas for rectangles be valid for squares or will formulas for squares be valid for rectangles?
Student: Formulas for rectangles will be valid for squares since all squares are also rectangles.
Teacher: Which of these area formulas should appear first then?
Student: The formula for rectangles because it will then be possible to deduce the formula for squares from it.
Teacher: Now let us consider the area propositions for triangles and trapezoids. Which one would you deduce first?
Student: Since I can divide a trapezoid into two triangles by drawing a diagonal, I would prefer to have the area proposition for triangles first.
Teacher: Let us now consider a rectangle and a parallelogram. Which should appear first in our logical order?

Note: The answer to this last question is not too obvious and the teacher may expect a lively discussion in which some of these points will be made:

It was easier to count the squares in a rectangle. Therefore, the rectangle should come first.

The rectangle is a more common shape. Therefore, it should come first.

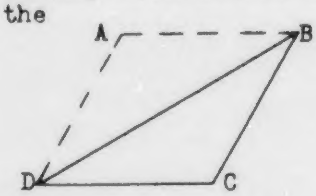
The rectangle is a subset of parallelograms. Therefore, the parallelogram should come first.

The teacher may now say that it is possible to make the deduction in either order. This order will depend upon our choice of a postulate. It seems more satisfying to start with the rectangle area proposition as the postulate, since it seems to be taking less for granted. (This conforms to the aesthetic impulse to assume as little as possible.)

Teacher: We have now agreed that in our logical order the area proposition for rectangles precedes those for squares and parallelograms. Similarly, the area proposition for triangles precedes that for trapezoid. Does this now give us a complete ordering?

Student: No. We don't know whether to have triangles before or after parallelograms or rectangles.

Teacher: May we deduce the area of a triangle easily from that of a parallelogram? Teacher draws the figure at the right on the blackboard.



Student: Yes. The triangle is half of the parallelogram.

Teacher: How should we arrange our four formulas in order, omitting the square?

Student: Rectangle, parallelogram, triangle, trapezoid.

There should be very little difficulty in developing the formal deductive proofs, except possibly for the parallelogram. This may be facilitated by the use of a paper model of a parallelogram from which the teacher cuts a right triangle from one end and moves it to the other end to form a rectangle.

Have the students consider the corollaries of the triangle area theorem. The formula for the equilateral triangle may be deduced. Have the students discuss the fact that the discovery of this formula by counting squares presents serious difficulties since $\sqrt{3}$ is irrational. The same difficulties will be found in the problem of discovering the formula for the area of a circle. This points up the advantage of deduction over experimentation in certain situations. It also demonstrates the predictive value of postulational thinking. The same discussion may be undertaken in proving by deduction the formula, $A=1/2\ ab \sin C$.

Have the students consider the trapezoid formula and how it may be viewed as a generalization of other formulas by letting the upper base become equal to the lower base (parallelogram) or shrinking to a point (triangle).

In assigning homework problems, the teacher should include deduction exercises concerning areas. (In classes where enrichment materials may be introduced, the topic of centroids and area transformations may be suitable.)

Lessons 6 and 7

Aim: To develop skill in the use of the area formulas.
To use trigonometric ratios in finding areas.
To use coordinates in finding areas.

Development:

Have the students use the area theorems to calculate the areas of figures for which the formula data are given directly. Next, include problems in which the data are insufficient, irrelevant or overabundant. The students may be asked to find the areas in such problems as the following:

In parallelogram ABCD, AC = 10" and AD = 5".
In triangle ABC, AC = 12" and the altitude on AB equals 10".
In trapezoid ABCD, (AB || DC), AB = 10", BC = 8", and CD = 7".
In triangle ABC, AC = 10", AB = 9", and the altitude on AB equals 9".
In trapezoid ABCD (AB || CD), AD ⊥ AB, AB = 10", BC = 6", CD = 7", and DA = 5".

The set of problems might also include finding areas of figures that can be dissected into rectangles, squares, parallelograms, triangles, and trapezoids.

The set of problems should also include some in which information is given that will help the student deduce needed formula data.

Find the area of a rectangle if its length is 12" and its diagonal is 13".
Find the area of $\triangle$ ABC if $\angle A = 30^\circ$ and the lengths of the included sides are 12" and 16".
Find the area of the isosceles trapezoid whose base angles are 45° and the bases are 8" and 20".

The set of problems should also include some that depend on algebraic techniques.

The area of a square is equal to that of a rectangle whose width is 3 units more than the side of the square, and whose length is 2 units less than the side of the square. Find the length of a side of the square.

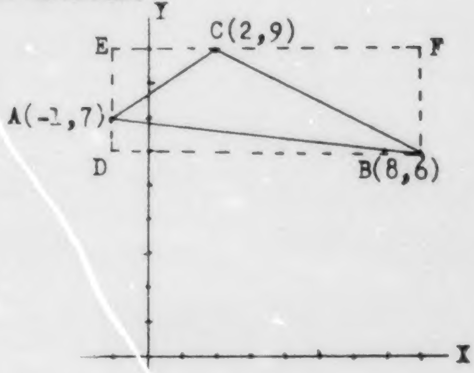
Prove that if the length of a square is increased by one unit, and its width is decreased by one unit, the rectangle thus formed will have one square unit less than that of the square.

Have students consider area problems involving trigonometric ratios.

Suggested exercises involving coordinates:

1. Plot A(-1,7), B(8,6), C(2,9) and find the area of $\triangle ABC$.

This type of exercise has proved useful in showing that the altitude must be perpendicular to the base. Students will finally suggest either of the two methods shown below. They should develop both methods.

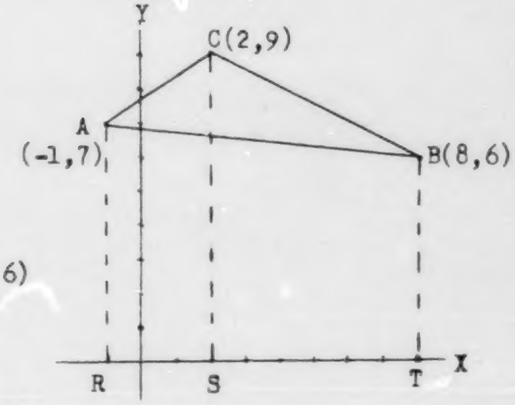


Acceptable student answer (type 1)

Draw lines through vertices forming rectangle EFBD.

$$\text{Area}_{ABC} = \text{Area}_{EFBD} - \left(\text{Area}_{ACE} + \text{Area}_{CFB} + \text{Area}_{BDA}\right)$$

$$= 9 \times 3 - (\tfrac{1}{2} \times 3 \times 2 + \tfrac{1}{2} \times 6 \times 3 + \tfrac{1}{2} \times 9 \times 1)$$
$$= 27 - (3 + 9 + 4\tfrac{1}{2})$$
$$= 27 - 16\tfrac{1}{2}$$
$$= 10\tfrac{1}{2}$$



Acceptable student answer (type 2)

Drop perpendiculars to the X-Axis.

$$\text{Area}_{ABC} = \text{Area}_{ACSR} + \text{Area}_{CBTS} - \text{Area}_{ABTR}$$

$$= \tfrac{1}{2} \times 3(7 + 9) + \tfrac{1}{2} \times 6(9 + 6) - \tfrac{1}{2} \times 9(7 + 6)$$
$$= \tfrac{1}{2} \times 3 \times 16 + \tfrac{1}{2} \times 6 \times 15 - \tfrac{1}{2} \times 9 \times 13$$
$$= 24 + 45 - 58\tfrac{1}{2}$$
$$= 69 - 58\tfrac{1}{2}$$
$$= 10\tfrac{1}{2}$$

2. Plot A(-3,2), B(5,-2), C(9,3), D(1,7). Why is ABCD a parallelogram? Find its area.

3. Triangles are drawn, two of whose vertices are (0,0) and (a,0). How does their area vary as:

   a. the third vertex moves on the line $y = c$
   b. the third vertex moves on the line $x = k$

4. Consider triangle A(0,0), B(a,0), C(b,c). How is the area affected:

   a. if a is doubled, and b and c remain constant
   b. if b is doubled, and a and c remain constant
   c. if a and c are doubled, but b remains constant

5. Find the area under the graph of equation $y = 2x + 7$ and above the X-Axis from $x = 0$ to $x = 8$. (optional)

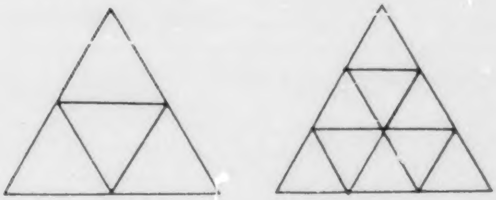
LESSON 8

Aim: to develop the theorem concerned with the ratio of areas of similar triangles

Development:

This lesson may be introduced with the story of a manufacturer who had a government order to stamp out a metal plate in the shape of a triangle. The price per plate was set at $2. After a time, the government asked for a larger plate of the same thickness in which each side was doubled and asked for a new price. The manufacturer suggested $4 per plate. Soon the manufacturer became aware that he was losing money. He called in a mathematician who informed him that he was using more than twice as much metal for each plate! How many times as much metal was he now using for each plate?

A simple illustration will elicit the answer that the triangular plate is four times as large as the original. Tripling the side should result in a triangle that is 9 times as large. Students are now ready for the generalization and the deduction may be done informally.



Exercises should include those in which sides or ratio of sides are given and areas or ratios of areas are to be found. Also, the areas or ratio of areas should be given and the sides or ratio of sides should be found.

It would be valuable for students to realize that the ratio of the squares of two numbers is the same as the square of their ratio.

Thus, instead of working with $\frac{6^2}{9^2}$ we may work with $\frac{6}{9}^{2}$ or $\frac{2}{3}^{2} = \frac{4}{9}$

Surprisingly, many students think that squaring the numerator and denominator of a fraction leaves its value unchanged! This shows the danger of teacher generalizations like, "What you do to the numerator of a fraction, you must do to the denominator."

UNIT TEST

AREAS

1. Arrange the area definitions and propositions in their logical order stating which is definition, postulate or theorem.

2. Deduce that the area of a trapezoid is equal to the product of one half its height and the sum of its bases.

3. Find the area of the following whose vertices are:

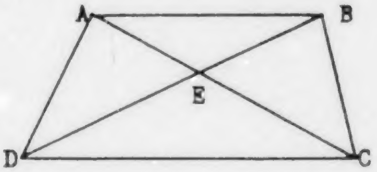
   a. Triangle ABC, A = (0,0), B = (3,4), C = (8,1).
   b. ABCD, A = (3,1), B = (7,1), C = (5,4), D = (1,4)
   c. ABCD, A = (-2,3), B = (4,3), C = (-3,-1), D = (4,-1)

4. In parallelogram ABCD, ∠A = 50°, AB = 12", AC = 20"

   a. find the length of the altitude from B to AC to the nearest tenth of an inch
   b. find the area of ABCD to the nearest square inch.

5. ABCD is a trapezoid with AB ∥ DC

   a. Deduce △ ADC = △ BDC
   b. Deduce △ AED = △ BEC



6. If △ ABC ~ △ A'B'C' and AB = 6", A'B' = 9", and area of △ ABC = 36 sq. in., find the area of △ A'B'C'.

XII.

REGULAR POLYGONS AND CIRCLES

Lesson 1

Aim: To develop the formula for the sum of interior angles of any polygon.
To develop the formula for the sum of the exterior angles

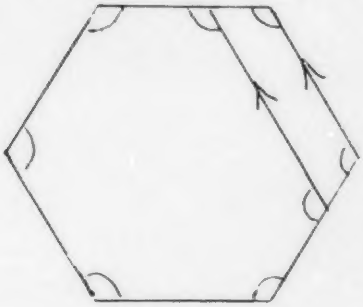
Development:

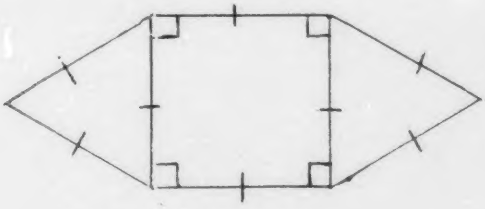
Have students draw polygons of 3,4,5,...n sides. Challenge them to suggest ways of figuring out the sum of the angles and tabulate the results. Students may suggest selecting a point within the polygon and connecting it to each vertex, thus forming n triangles the sum of whose angles is 360° more than the sum of the angles of the polygon. Other students may suggest drawing diagonals. This group will discover the disadvantage of criss-crossing diagonals and will improve their procedure by limiting the diagonals to those coming from a single vertex. Each side of the polygon, with the exception of the two meeting at the single vertex, now becomes the base of a $\triangle$. The students should then be able to make the deduction which generalizes these results.

Apply the generalizations to exercises including the special cases of equiangular polygons and their special formulas.

When we discovered that the sum of the angles of (for instance) a decagon is 1440°, were we able to determine the number of degrees in each angle? Could we determine the number of degrees in each angle if we knew that all the angles were equal? Students often believe that an equiangular polygon is equilateral or that an equilateral polygon is equiangular.

Show that a polygon could be equiangular without being equilateral (square vs. rectangle) or that it could be equilateral without being equiangular (square vs. rhombus). Convince pupils that this is also true of such polygons as hexagons by diagrams such as the following:





Define a regular polygon and ask which is the regular triangle and regular quadrilateral. Note the frequency of the regular polygon in design - rose windows, tiles. Ask why nuts are more often hexagonal than square.

Compute each angle of the various regular polygons. Note that each angle of an equiangular hexagon is always 120°. Is each angle of an equiangular octagon more or less than 120°? Of an equiangular decagon? Calculate the exterior angle for each of the above. Students will note that as the number of sides of an equiangular polygon increases, each interior angle increases and each exterior angle decreases. Now consider the sum of the angles again. Obviously the sum of the interior angles for any polygon increases as the number of sides increases. Does the sum of the exterior angles decrease? Deduce the surprising fact that the sum of the exterior angles is constant! This will excite the imagination as students envision more and more sides, therefore more and more exterior angles with a constant sum.

LESSON 2

Aim: to develop ideas concerned with similar polygons

Development:

Have the students review the definition of similar figures:

equality of corresponding angles
equality of ratios of corresponding sides

The class should then investigate the following:

Are any two squares similar?
Are any two rhombuses similar?
Are any two regular pentagons similar?
Are any two rectangles similar?
Are any two equiangular pentagons similar?

Have the students start with one side and construct a quadrilateral similar to a given quadrilateral. This should lead to the theorem:

> If two polygons can be divided into pairs of similar triangles similarly placed, then the polygons are similar.

The converse of the above theorem should also be considered.

LESSON 3

Aim: to study the construction relationships between circle and regular polygon

Development:

Have the students give some examples of the frequent occurrence of regular polygons and circles in design and industry.

Experiment: Inscribe a square in a circle.

The outcome of the experiment and the discussion concerning it should lead to the theorem:

> Regular polygons of n sides may be inscribed in a given circle if the circle can be divided into n equal arcs.

Have the students consider the problem of inscribing regular polygons of 8, 16, 32...sides.

The problem of inscribing a regular hexagon should be related to the fact that a regular hexagon has 6 equilateral triangles meeting at the center. The problem of inscribing regular polygons of 3, 12, 24...sides should then be considered.

The problem of circumscribing a regular polygon of n sides about a given circle should be related to the basic problem of dividing a circle into n equal parts. It will be necessary to review the construction of a tangent to a circle at a point on the circle.

LESSON 4

Aim: to deduce that a circle can be circumscribed about (or inscribed in) any regular polygon
to introduce the concepts of center, radius, apothem, and central angle of a regular polygon

Development:

Have the students review the circumscription of a triangle and then consider the cases of a rectangle and a rhombus. A discussion of these will lead quickly to the need for locating the center of the perpendicular bisectors of the sides of polygon.

The following discussion may now take place:

Teacher:	Consider a regular polygon of an indefinite number of sides. If the polygon has a center of a circumscribing circle, how can we locate it?
Student:	Construct the perpendicular bisectors of two adjacent sides. The point where the bisectors meet should be the center.
Teacher:	What would be the radius of the circle?
Student:	The distance between the point of intersection and a vertex.
Teacher:	How do we know that the distance from the point of intersection to any vertex will be the same?

A discussion of the teacher's last question should lead to the theorem:

> A circle can be circumscribed about, or inscribed in, any regular polygon.

The center of the regular polygon can now be defined as the center of the circumscribed (or inscribed) circle. The radius, apothem, and central angle of a regular polygon may now be defined. The students should note that a regular polygon of n sides has exactly n radii, n apothems, and n central angles.

Have the students discover that the radius of a regular polygon bisects its angle, and the apothem bisects the side and central angle.

Have the students do exercises in which the length of a side of a regular polygon (3, 4, or 6 sides) is given and the radius, apothem, and central angle are to be found.

LESSON 5

Aim: to consider more difficult exercises involving side, radius, apothem, and central angle of a regular polygon
to deduce that the area of a regular polygon is equal to one half the product of its perimeter and its apothem
to relate the area of a regular polygon to the area sequence

Development:

Have the students consider exercises involving regular polygons in which the radius or the apothem, or the side, or the perimeter is given in addition to the number of sides of the polygon. They should have practice in deducing the number of degrees in each angle of the right triangle formed by a radius, adjacent apothem, and side of the polygon.

Suggested exercises:

1. The radius of a regular decagon is 10. Find the apothem, side, and perimeter.
2. The apothem of a regular octagon is 20. Find the radius, side, and perimeter.
3. The perimeter of a pentagon is 100. Find the area of the isosceles triangle formed by two adjacent radii. Find the area of the pentagon.

The last exercise may serve as the basis for the development of the deduction of the area theorem for a regular polygon. This should be a formal deduction since the theorem may be called for on the Regents examination.

Students should review the area sequence (postulate - area of rectangle; theorem - area of a parallelogram; theorem - area of a triangle) and notice that the area of a regular polygon theorem follows that of the area of a triangle theorem.

The assignment should include the problem of finding the ratio of the perimeter of a regular polygon to twice the radius. If the number of sides of the regular polygon is n and the radius is r, assign the problems so that the students in each row have the same value of n and different values of r. For example, the students in the first row will take regular hexagons, and the first student will take $r = 1$, the second student will take $r = 2$, and so on. The students in the second row will take $n = 12$, those in the third row will take $n = 24$, and so on. Have one row take $n = 180$, since this is the largest number of sides one could take without involving an angle less than 1° (note that our trigonometry tables permit entries to nearest degree only).

LESSON 6

Aim: to develop the ratio (π) of the circumference of any circle to its diameter
to develop an appreciation of the nature of π
to postulate that whatever is true of a regular polygon, and which does not depend upon the number of sides of the polygon, is also true of the circle
to find circumferences and lengths of arcs

Development:

The lesson may begin with reports of the results of the homework exercises. The students in each row will discover that the ratio of the perimeter to twice the radius is the same as long as the number of sides is the same. They will conclude that the ratio is constant for any given n. This may be deduced informally.

The circle may now be introduced as the limiting shape of a regular polygon as the number of sides increases. Hence, we postulate that:

> Whatever is true of a regular polygon, and which does not depend upon the number of sides of the polygon, is also true of the circle.

An application of the postulate gives us $\frac{C_1}{C_2} = \frac{d_1}{d_2}$.

This implies that $\frac{C_1}{d_1} = \frac{C_2}{d_2}$. The student should be led to realize that this proposition means that the ratio of the circumference of any circle to its diameter is the same or constant. Have the students refer again to their homework results:

Teacher: How can we arrive at an approximation of this constant value for $\frac{C}{d}$?
Student: We can arrive at the approximation by observing the constants for n = 6, 12, ... 180.
Teacher: Which of these would be best for our purpose?
Student: The case in which we have the largest possible number of sides.

Since the ratio of the circumference of any circle to its diameter is a constant, we give that constant a name. We have no numeral to designate this number, therefore we choose the symbol π for it. π, the Greek letter "p", was chosen in the 18th century because it is the initial letter of "periphery," the word then used to designate the circumference.

Thus, $\pi = \frac{c}{d}$ so that $c = \pi d$, (or $c = 2\pi r$).

Students may be confused into thinking π is rational because it is defined as the ratio $\frac{c}{d}$. The fact is that c and d cannot both be integers. Give exercises like, "If the circumference of a circle is 10, find the radius." Contrasting this with, "If the radius is 10, find the circumference," will clarify this point.

The value of the ratio for $n = 180$ is reported as 3.15. The teacher may offer the information that for $n = 200$, the ratio $= 3.1412$. The first good approximation of the value of the ratio was found by Archimedes. He found its value to lie between 3 1/7 and 3 10/71.

The value of the ratio cannot be [illegible] a ratio of integers. The ratio is designated by the symbol π and is [illegible] to be an irrational number. Thus, $\frac{C}{2r} = \pi$ and $C = 2\pi r$.

Have the students use this in finding circumferences and length of such arcs as 180°, 90°, 45°, 60°, and 30°. Answers, for the most part, should be left in terms of π. Students should estimate answers using 3 for π.

Lesson 7

Aim: To deduce the area of a circle ($A = \pi r^2$)
To trace back the area sequence.

Development:

Have the students consider a regular polygon inscribed in a circle. As the number of sides of the polygon increases, it assumes more and more the shape of the circle. The students should realize that (1) the apothem increases and approaches the radius of the circle as a limit and (2) the perimeter increases and approaches the circumference as a limit. Therefore, the formula for the area of a regular polygon ($A = \frac{1}{2}ap$) becomes for the circle $A = \frac{1}{2}r\,C$ or $\frac{1}{2}r\,2\pi r = \pi r^2$.

Suggested exercises:

1. The area of a circle is 25π. Find the radius and circumference.

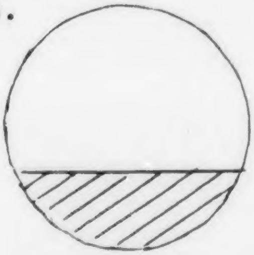
2. The circumference of a circle is 14π. Find the radius and area.

Teacher: What was the original postulate of our area sequence?

This question will spark a reexamination of the entire area sequence ending with the area of the circle.

Lesson 8

Aim: To find areas of sectors and segments.
To solve area problems involving composite figures.



Development:

Tell the class that the area of such a piece of a circle as the shaded portion of the diagram arises in problems involving pipes or horizontal tanks partially filled with water. Challenge them to suggest how to find such an area.

Students will suggest drawing the radii, and this suggestion should lead to definitions for both sectors and segments.

Compare the area of the sector as a fractional part of the circle to the corresponding notion of the length of an arc as a fractional part of the circumference of the circle.

Discourage memorization of unnecessary formulas. The area of a minor segment should be attacked each time it is met as the area of a sector minus that of a triangle. Note that the $\frac{1}{2}ab \sin C$ method is usually the shortest method for getting the area of the triangle in this context.

UNIT TEST

REGULAR POLYGONS and CIRCLES

1. State the theorems, in order, leading from the postulate, "The area of a rectangle is the product of its base and altitude," in the sequence which concludes with the deduced theorem, "The area of a circle is $\frac{1}{2}$ the product of its circumference and radius."

2. A regular decagon is inscribed in a circle of radius 4 inches.

   a. Find the apothem of the decagon.
   b. Find the side of the decagon.
   c. Find the perimeter of the decagon.
   d. Find the area of the decagon.

3. Find the number of degrees in the central angle of a sector whose length of arc is 3π in a circle whose radius is 6.

4. The area of a circle is 40 square inches. Find the area of a circle whose radius is 3 times as large.

5. Find the area of a segment whose chord is 8 inches long and whose arc contains 120°.

6. A regular pentagon is cut out of a circular piece of sheet metal whose radius is 6 inches. How many square inches of metal are wasted?

XIII.

LOCUS EXTENDED

LESSON 1

Aim: to review the idea of locus
to review two locus theorems
to deduce that the locus of points equidistant from two given parallel lines is a line parallel and midway between the two given lines
to apply locus theorems to exercises

Development:

Have the students plot the locus of points satisfying x = 5. Have them also plot the locus of points satisfying x = -1. What does locus mean? The students will recall the definition of locus as the set of points satisfying a given condition.

The students should recall the two locus theorems previously listed:

> The locus of points within an angle and equidistant from the sides of the angle is the bisector of that angle.
>
> The locus of points equidistant from two given points is the perpendicular bisector of the line segment joining the two points.

What is the locus of the points equidistant from x = 5 and x = -1? The locus is a line parallel to x = 5 and x = -1 and midway between them, namely x = 2. (Review midpoint formula.)

Have the students recall that a locus theorem is a combination of a subsidiary theorem and its converse. The subsidiary theorem and its converse which arise from the class exercise are:

> Every point on the line x = 2 is equidistant from x = 5 and x = -1. If a point is equidistant from x = 5 and x = -1, it lies on the line x = 2.

Have the students state the locus theorem of points equidistant from two parallel lines, without the use of coordinate geometry, and add it to their list of locus theorems.

Suggested exercises:

The students should be led to discover loci propositions by making several colored dots to represent members of the required set of points.

1. Teacher: What is the locus of the points 4 units from the origin?
   Student: It seems to be a circle of radius 4 with center at (0,0).

   Enrichment: Teach the algebraic technique of letting (x,y) be any point on the locus. Derive the equation of the locus. For example, let (x,y) be any point four units from (0,0).

   Distance formula: $\sqrt{x^2 + y^2} = 4$

   $$x^2 + y^2 = 16$$

   What would this locus be if the discussion were concerned with space rather than with the plane?

2. How many points are four units from the origin and also equidistant from $x = 5$ and $x = -1$?

3. Locate all points "d" units from a given point A and equidistant from two parallel lines m and s. Discuss all possible solution sets.

LESSON 2

Aim: to investigate other locus theorems
to practice a technique for handling any locus problem

Development:

Have the students consider a problem such as the following:

> We wish to locate campsites two miles from the straight road m, and ten miles from the town S on road m. How many such campsites are there?

What new locus theorem is needed for the first requirement - the campsites are to be two miles from the road? The students should be led to see that the locus of points a given distance from a given line is two lines parallel to the given line and the given distance from it.

What locus theorem is needed for the second requirement - the campsites are ten miles from the town S? How many points satisfy both conditions?

Enrichment: What would the locus be if the discussion were in space instead of in the plane?

Suggested exercises:

1. Have the students apply the locus theorem of points a given distance from a given line to coordinate geometry. For example, write the equation of the locus of points two units from $y = 7$. The students will see that the result

is two lines, $y = 9$ and $y = 5$, because neither equation alone contains all the members of the required set.

2. Investigate other loci by the colored-dot method. For example, find the locus of the points four units from a circle of radius seven. What would the locus be if the discussion were in space?

   Note: Encourage students to sketch solutions involving several loci by keeping the given facts in ink and each locus in a different color.

3. Enrichment:

   a. Derive the equation of the locus of the points 4 units from (2,3).
   b. Show that the equation of the locus of all points whose distance from (2,3) equals their distance from $y = -1$ is in the form $y = ax^2 + bx + c$. Use colored dots and sketch the locus. This locus, a parabola, is very important in reflectors and telescopes.

Lesson 3

Aim: To apply locus theorems to the problems of circumscribing a circle about a triangle and inscribing a circle within a triangle.

Development:

Have the students draw any triangle ABC and consider the problem of circumscribing a circle around ABC.

The question will arise as to where is the center of the circle. This will lead the students to recall the method of finding the center of a broken wheel. They will agree that two loci are sufficient to locate the point which is equidistant from all three vertices. Deduce this orally using the locus of points equidistant from two given points. Complete the construction.

Have the students next consider the problem of drawing a circle within triangle ABC tangent to the sides of the triangle (sketch). We call this an inscribed circle.

Teacher:	Will the inscribed circle havethe same center as the circumscribed circle?
Student:	No. The center of the inscribed circle must be equidistant from the sides.
Student:	Then it must be the intersection of the angle bisectors. Note: Have students recall the locus theorem of points within an angle equidistant from the sides.
Teacher:	When we know the center, what else must we know in order to construct the circle?
Student:	We need the radius. It is the distance from the center to a side.
Teacher:	What construction must we make in order to find this distance?
Student:	Drop a perpendicular from the center to a side.

Have the class complete the constructions. Have them consider:

When will the circumcenter and incenter coincide?
When will the circumcenter be in, on, or outside the triangle?

Enrichment: Find the coordinates of the circumcenter of triangle A (2,1), B (5,7), C (7,6). Deduce what kind of triangle it is from your result.

Given isosceles right triangle with hypotenuse $5\sqrt{2}$. Find the inradius.

UNIT TEST

LOCUS EXTENDED

1. Locate by construction all points 3 units from a given point P and equidistant from two parallel lines if the lines are 1 unit apart and P lies on one of the lines.
2. Derive the equation of the locus of all points 5 units from (0,0) and draw it.
3. Quote a locus theorem, and the subsidiary theorem and converse of which it is made up.
4. Show by construction in a scalene triangle that the circumcenter and incenter are two different points. Discuss special cases.

XIV.

INEQUALITIES

Lesson 1

Aim: To learn how to form the inverse of an if-then statement.
To recognize that a statement need not have the same truth value as its inverse.
To compare the inverse and the converse of a statement.
To learn how to form the contrapositive of an if-then statement.
To recognize that a statement and its contrapositive have the same truth value.
To introduce inequalities by forming the inverse and contrapositive of statements of equality.

Development:

The topic may be introduced by asking, "How can we form new propositions from known propositions?"

Have the students recall how to form the converse of an if-then statement. Have them apply it to, "If two angles are right angles, then the two angles are equal."

They should recall that converses are introduced to help formulate new statements which are to be tested as possible theorems. They should be led once again, by means of real-life and geometric examples, to realize that "arguing from the converse" is a type of fallacious reasoning, since a statement need not have the same truth value as its converse.

Show the class that another transformation of an if-then statement is the inverse of the statement. The inverse is formed by replacing both the hypothesis and the conclusion by their negations. The result is a new statement to be tested as a possible theorem. The inverse of the above statement about angles is, "If two angles are not right angles, they are not equal." This is a statement of inequality and it happens to be false.

Have the students consider a few more inverses of theorems and ascertain their truth values. Include real life statements as well as geometric statements, for example: "If you are a citizen, then you may vote," and "If you live in New York City, then you live in the United States." Have the students conclude that "arguing from the inverse" is another type of fallacious argument.

Re-emphasize the fact that just as each time that a proposition was discovered by experimentation its truth had to be established by deduction, so also each time a proposition is formulated as a converse or as an inverse of a given statement, the truth of the resulting statement must also be established.

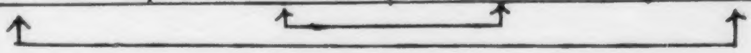
Now pose the question of formulating the inverse of the converse of a given if-then statement. Use the examples previously considered, as well as new examples, and have the students compare the truth values. Using, "If A, then B", show that the converse of the inverse is the same as the inverse of the converse. This statement, If not B, then not A, is called the contrapositive of If A, then B. Elicit the generalization: an if-then statement and its contrapositive have the same truth values.

An additional result, if desired, is the generalization: the converse and the inverse of an if-then statement have the same truth values - that is, they are either both true or both false.

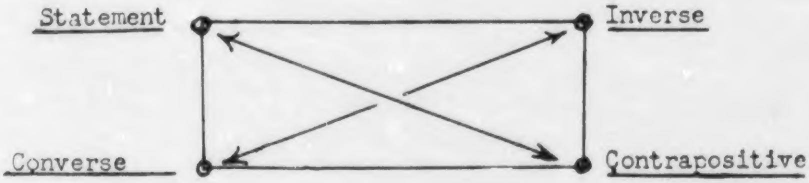
Make the presentation of these logical notions as concrete as possible by the use of many illustrations and by giving much practice. In particular, if statements of equality are used, the new propositions formed as inverses or contrapositives will deal with inequalities. For example: "If two sides of a triangle are equal, then the two angles opposite these sides are equal"; "In a circle, if two chords are equal, then their arcs are equal"; and so on. Note also that such statements of inequality can be made more precise by introducing the notion of "greater than" or "less than". This will be done in later lessons.

The complete discussion may be summarized in the following tabulation, in which the truth values "True" and "False" are symbolized respectively by "T" and "F".

	Statement	Converse	Inverse	Contrapositive
Logical Form	If p, then q	If q, then p	If not p, then not q	If not q, then not p
Set of Possible Truth Values	T	T	T	T
	T	F	F	T
	F	T	T	F
	F	F	F	F



An alternative way to represent these logical interrelationships is the following diagram.



A better class may consider, as enrichment, the formation of partial inverses of an if-then statement. For example, "If a line passes through the center of a circle and is perpendicular to a chord, then the line bisects the chord" has the partial inverses: "If a line does not pass through the center of a circle, then the line is not the perpendicular bisector of a chord," and "If a line is not perpendicular to a chord, then the line does not pass through both the center of the circle and the midpoint of the chord."

Lesson 2

Aim: To review the meaning and notation of the inequality relations symbolized by $\neq$, $>$, and $<$.

To begin listing inequality postulates as a step in the development of another miniature postulational system.

To elicit the postulates:

for any two quantities, either the first is greater than the second, or else the first is equal to the second, or else the first is less than the second

the whole is greater than any of its parts

in any statement of inequality a quantity may be replaced by its equal

if the first of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.

To deduce and apply the theorem:

an exterior angle of a triangle is greater than either nonadjacent interior angle.

Development:

Pose the problem, "How does an exterior angle of a triangle compare in size with an adjacent interior angle? with a nonadjacent (remote) interior angle?"

One result will be the proposition, "An exterior angle of a triangle is greater than either nonadjacent interior angle." Ask, "How can we deduce this?"

At this point the more alert students may assert that the exterior angle has already been shown to be equal to the sum of the two nonadjacent interior angles, and so must be greater than either one. Remind the class, however, that whenever deduction is to be used in a new area of mathematics, additional definitions and postulates are necessary. This is here the case, since we are now going to deal with statements of inequality instead of statements of equality.

Recall and re-emphasize the fact that equality of line segments or equality of angles means equality of the positive real numbers which are their measures (number of units of linear measure or number of units of angular measure).

Define "a is greater than b" to mean "there is some positive number, p, such that $a = b + p$." Illustrate this definition by using the coordinates on a line to show that "a is greater than b" means that the point whose coordinate is a is to the right of the point whose coordinate is b."

Review or re-teach the notations: $a > b$, $a < b$, $a \neq b$. Show also how to read statements in which these symbols are used. In particular, emphasize that "$a < b$" is equivalent to "$b > a$", and that "$a \neq b$" is equivalent to "either $a > b$ or else $a < b$." Illustrate these notions and notations with particular line segments and particular angles.

Review the "trichotomy property", that for any two real numbers (quantities) a and b, one and only one of the following relations is true: $a > b$, $a = b$, $a < b$. Adopt the statement of this property as the first postulate in a list to be used for the purpose of deducing theorems about inequalities.

Lead the class to see that from the definition of "greater than" a reasonable assumption is, "The whole is greater than any of its parts." Adopt this statement as the second postulate in the list of inequality postulates. Illustrate the postulate with line segments and with angles.

Now have the students deduce the "greater than" theorem for the exterior angle by using the equality theorem and the postulate that the whole is greater than any of its parts.

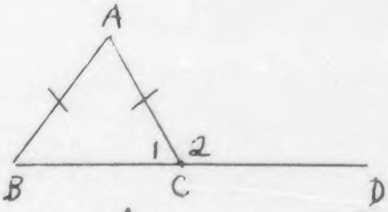
Apply the new theorem to exercises. As this is done, the class will find that corresponding to the "substitution postulate" for equalities, it is useful also to have the postulate: in any statement of inequality, a quantity may be replaced by its equal. Have the students add this postulate to their list.

Another needed postulate is the "transitive property" of the "greater than" relation: if $a > b$, and $b > c$, then $a > c$. This property may be elicited by comparing the heights of three students. Have the students formulate a generalization of the transitive property and add it to the set of postulates.

Suggested exercises:

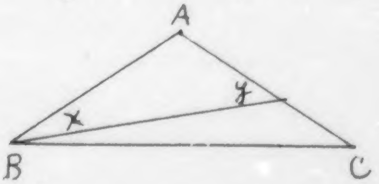
1. Given: $AB = AC$

   Deduce: $\angle 2 > \angle 1$



2. Given: $AB = AC$

   Deduce: $\angle y > \angle x$



3. Enrichment: Without using any dependence on the parallel postulate, deduce: An exterior angle of a triangle is greater than either remote interior angle. Use the necessary assumptions of betweenness.
   Note: The theorem, "The exterior angle of a triangle is equal to the sum of the two remote interior angles", is a consequence of the parallel postulate.

4. Enrichment: Deduce the "transitive property" postulate from other postulates and the definition of "greater than".

Lesson 3

Aim: To complete the set of inequality postulates.

To postulate:

- a smaller quantity may be made part of a larger one
- if equals are added to (subtracted from) unequals, the results are unequal in the same order
- if unequals are multiplied (divided) by positive equals, the results are unequal in the same order
- the sum of two sides of a triangle is greater than the third side

To deduce and apply the theorem:

- if two sides of a triangle are unequal, then the angles opposite these sides are unequal, and the greater angle lies opposite the greater side.

Development:

Ask, "What is the inverse of: If two sides of a triangle are equal, then the angles opposite these sides are equal. Is the resulting new statement true or false?" Have the students make the new statement more precise by using "greater" to clarify "unequal". Pose the problem of deducing: "If two sides of a triangle are unequal, the angles opposite these sides are unequal, and the greater angle lies opposite the greater side."

A possible procedure is to deduce the theorem in two stages: a preliminary problem, and the main problem. For the preliminary problem, have the students consider Diagram 1 of the following diagrams, with $AD = AC$ given. Have them deduce that angle y is greater than angle B, by making use of postulates and the exterior angle inequality theorem.

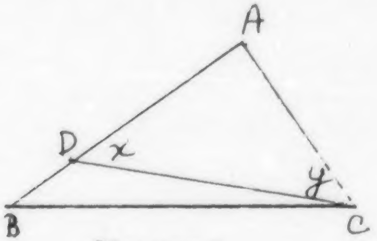


Diagram 1

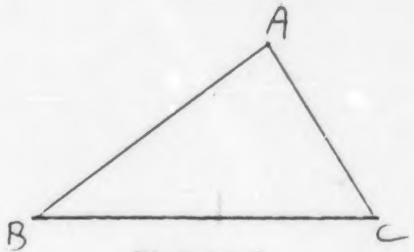


Diagram 2

Continue with the main problem, referring to Diagram 2: if in triangle ABC it is given that $AB > AC$, deduce that $\angle C > \angle B$. Make use of a point D on AB such that $AD = AC$, and then proceed as with the preliminary problem. In order to be able to assert that there is such a point, we need to add to our list of inequality postulates: "a smaller quantity may be made part of a larger one."

Have the class realize that in order to apply the inequality theorems effectively, it will be necessary to complete the set of inequality postulates. Try to elicit from the students the inequality postulates that are analogs to the equality postulates of addition, subtraction, multiplication and division.

This might be done by having the students consider: $14 > 8$
$2 = 2$

Note that, in each of these postulates, the equal quantities operate on the unequal quantities. Consider also the cases of inequalities operating on equalities, especially subtraction and division.

The students should also examine situations in which no conclusive generalization can be made. For instance, if $a > b$ and $c < d$, nothing can be said about the relative sizes of $a + c$ and $b + d$.

A better class, for enrichment, might consider the inequality postulates for multiplication and division when the equality is between negative numbers. However, such postulates are not needed for purely geometric inequalities.

Discuss the apparent assumption about distances made by a pedestrian who jaywalks. Have the class adopt and add to the list of inequality postulates the corresponding "triangle inequality": the sum of two sides of a triangle is greater than the third side.

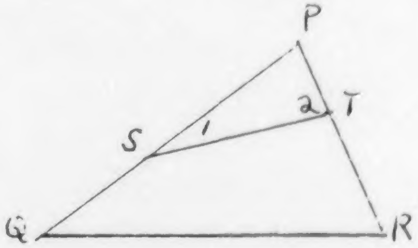
Have the class review and list: ways to deduce that one line segment is greater than another, and ways to deduce that one angle is greater than another.

Suggested exercises:

1. Given: $PQ > PR$

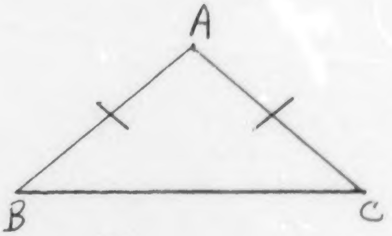
   $SQ = TR$

   Deduce: $\angle 2 > \angle 1$



2. Given: $AB = AC$

   Deduce: $AB > \frac{1}{2}BC$

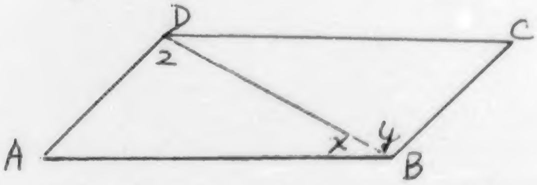


3. Given: ▱ ABCD

   diagonal BD

   $AB > BC$

   Deduce: $\angle y > \angle x$



Note: What does this exercise tell us about the diagonals of a parallelogram which is not a rhombus?

4. In Exercise 3, what can be deduced if $AB < BC$? if $AB = BC$?

5. If two sides of a triangle are unequal, prove that one angle of the triangle is greater than 60°.

6. Enrichment: If $a > 0$, does it follow that $4a > 3a$? Suppose $a < 0$?

Lesson 4

Note: While the formal treatment of the indirect proof has been deferred until the unit on inequalities, informal indirect proofs have been used earlier in the guide; e.g. the theorem, "the line perpendicular to a radius at its outer extremity is a tangent to the circle," has been proved by the indirect method (See Lesson 7 of VII on circles). The teacher might well have introduced a formal treatment of the indirect method of proof at one of these earlier points. If this is done, it is advisable to use indirect reasoning, wherever feasible, throughout the year. It is a method which the students will use frequently throughout higher mathematics.

Aim: To develop the method of elimination as an indirect method of deduction
To postulate:
  if a supposition leads to a contradiction, then the supposition is false
  if all but one of all the mutually exclusive possibilities for the conclusion of a statement are false, then the only remaining possibility must be true
To deduce, by the method of elimination, the theorem:
  if two angles of a triangle are unequal, then the sides opposite these angles are unequal and the greater side lies opposite the greater angle
To apply the method of elimination to prove original exercises.

Development:

Have the class recall the theorem: "If two angles of a triangle are equal, then the sides opposite these angles are equal." Ask the students to form the inverse and investigate its truth value.

Lead the students to see that the statement, "In triangle ABC, if $\angle B \neq \angle C$, then $AB \neq AC$", can be deduced at once if we adopt as a postulate: "If a supposition leads to a contradiction, then the supposition is false." For, if we suppose $AB = AC$, this would mean $\angle B = \angle C$, which contradicts $\angle B \neq \angle C$.

At this point an alert student may point out that the proof of this inequality theorem follows at once as the contrapositive of a theorem on equality.

Have the students consider the more precise statement, "In triangle ABC, if $\angle B > \angle C$, then $AC > AB$." In order to use the "contradiction postulate" to deduce this statement, we must make more than one supposition. To illustrate the procedure, have the students consider the case in which Joe is either at home, at Henry's home, or at the movies. We phoned Joe's home and learned that Joe is not there. We phoned Henry's home and learned that Joe is not there either. What can we deduce from this information?

Discuss the difficulty of listing all the possibilities in a real situation. Discuss the frequency of the use of this indirect method as a method of deduction in medical diagnosis, detective stories, and mechanical trouble shooting.

This type of indirect method may be called "the method of elimination." From the above, abstract the structure:

> List all the mutually exclusive possibilities for the conclusion in a proposition.
> Let A, B, and C be the possibilities with A the one we are trying to deduce.
> If supposition B, then a sequence of steps follows ending in a contradiction.
> Therefore B is eliminated, by the "contradiction postulate".
> If supposition C, then as for supposition B, a sequence of steps ends in a contradiction, and C is eliminated.
> Therefore A is true. As a reason, we need the postulate, "If all but one of all the mutually exclusive possibilities for the conclusion of a statement are false, then the only remaining possibility must be true." We may abbreviate this: "Only remaining possibility."

The following teacher-student interchange illustrates the use of this method of elimination as well as another possible approach to introducing the present inequality theorem:

Teacher: We have deduced the theorem that in a triangle, the greater angle lies opposite the greater side. What is the converse of this theorem?
Student: In a triangle, the greater side lies opposite the greater angle.
Teacher: Decide by experimentation whether the converse seems to be true. (The students make triangles with unequal angles and measure the opposite sides.)
Teacher: We are going to use the method of elimination to deduce this theorem. What is the first step in the procedure?

Student: List all the possibilities for the sides b and c.
Teacher: What are all the possibilities in this case?
Student: All the possibilities are: $b = c$, $b < c$, $b > c$. (For two quantities, either the first is greater than the second, or else the first is equal to the second, or else the first is less than the second.)
Teacher: What should be the second step in the deduction?
Student: Suppose $b = c$, then $\angle B = \angle C$. (If two sides of a triangle are equal, then the angles opposite them are equal.)
Teacher: What is our next step in the deduction?
Student: The angles are unequal; that is, $\angle B > \angle C$. (Given.)
Teacher: Therefore, what can we say?
Student: The conclusion "$b = c$" is false, and may be eliminated. (If a supposition leads to a contradiction, it is false.)

As this teacher-student interchange continues, the next possibility, $b < c$, is also shown to lead to a contradiction, and is therefore eliminated. The final step may then be written in the form: $b > c$. (Only remaining possibility.)

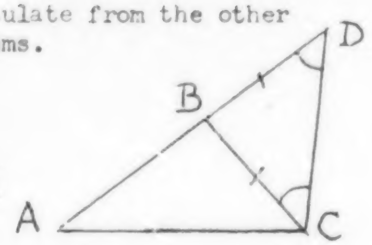
Suggested exercises:

1. In triangle ABC, if $AB = AC$, and D is any point in BC, deduce that $AB > AD$.
2. By the method of elimination, deduce that the hypotenuse of a right triangle is greater than a leg of the triangle.

3. Enrichment: Deduce the triangle inequality postulate from the other inequality postulates and the inequality theorems.

Suggested method:

Suppose AC is the longest side of the triangle. Extend AB to D so that $BD = BC$. Then show that $AD > AC$.



UNIT TEST

INEQUALITIES

1. In $\triangle ABC$, $\angle A = 60^\circ$, $\angle B = 65^\circ$. Name the longest side of the triangle.

2. From the two statements: $8 > 5$, and $3 = 3$, one may deduce the statement $8 - 3 > 5 - 3$. State the postulate that justifies the conclusion.

3. Give a brief argument justifying the statement: "The side opposite the obtuse angle in a triangle is the largest side of the triangle."

4. Deduce that the sum of the diagonals of a parallelogram is less than the perimeter.

5. Enrichment: Deduce by a direct method: If two angles of a triangle are unequal, the sides opposite these angles are unequal and the greater side lies opposite the greater angle.

XV.

INTEGRATION OF PLANE AND SOLID GEOMETRY

This year course in Tenth Year Mathematics is the development of a postulational system for two-space. The same course can well be taught to honor classes as a study of the postulational development of both two-space and three-space geometry.

Suggestions for accomplishing this integration of plane and solid geometry:

1. When studying perpendicular lines in two-space, introduce perpendicular lines and planes in three space. For example, in the unit on congruence, the class can prove: "If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to their plane."

2. When studying parallel lines in two-space, introduce parallel lines and planes in three-space. For example, the class can prove: "Two lines parallel to the same line are parallel to each other." Contrast this with such false statements as "Two lines perpendicular to the same line are parallel." Throughout the year, the class should decide which theorems in two-space hold in three-space.

   In the non-Euclidean geometry discussion, bring in the geometry on the surface of the sphere as a model of Riemannian geometry.

3. In introducing locus, extend the concept to sets of points in three-space. For example, the locus of points equidistant from two given points is the plane which is the perpendicular bisector of the line segment joining them.

4. In quadrilaterals, we assumed all the figures were plane. Show that a quadrilateral can be a skew figure, but that parallelograms and trapezoids are plane.

5. The locus definition of the circle can be extended in three-space to the sphere. Students can prove that the intersection of a sphere and a plane is a circle and study great circles and small circles.

6. In coordinate geometry, introduce the one-to-one correspondence between points in space and triples of real numbers. Equations of planes parallel to the coordinate planes are appropriate here.

   As soon as the distance formula has been developed for two-space, it may be extended to three-space. Equations of cylindrical surfaces and spheres are easily discussed.

7. After the Pythagorean Theorem and trigonometry, many numerical problems with spheres and polyhedra are easily understood.

8. In the unit on areas, surface areas of three-dimensional figures and their volumes will arise. Students have studied volume formulas in grade 8.

9. In connection with similar polygons, the relationship between the ratio of sides and areas can be extended to corresponding situations involving edges, surfaces and volumes of similar solids. Regular polygons should lead to a discussion of the five regular polyhedra.

10. With inequalities, basic inequalities in three dimensions, such as the face angles of the polyhedral angle, provide natural extensions.

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